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Newton polytopes and tropical geometry

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Abstract. The practice of bringing together the concepts of 'Newton polytopes', 'toric varieties', 'tropical geometry', and 'Gröbner bases' has led to the formation of stable and mutually beneficial connections between algebraic geometry and convex geometry. This survey is devoted to the current state of the area of mathematics that describes the interaction and applications of these concepts.

Bibliography: 68 titles.

Keywords: family of algebraic varieties, Newton polytope, ring of conditions, toric variety, tropical geometry, mixed volume, exponential sum.

Contents

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1. Introduction

This survey is devoted to the current state and interrelations between various branches of algebraic and convex geometry, namely the theory of Newton polytopes, geometry of toric varieties, and enumerative tropical geometry, as well as topics in valuation theory and Gröbner bases. We briefly recall what this is all about.

1.1. Newton polytopes. Let V be an irreducible algebraic variety whose points $\varphi \in V$ parameterize an algebraic family of varieties N_φ. Then the simplest discrete invariants of N_φ (the number of components, dimension, degree, Euler characteristic, genus, and so on) turn out to be constructive functions of φ on V (that is, finite linear combinations of the characteristic functions of algebraic subvarieties of V). In particular, every such an invariant takes the same value on almost all elements of the family (that is, for all φ in a Zariski open subset of V), which is referred to as the value of the given invariant at a generic element of the family.

In particular, a family of 'polynomials with indeterminate coefficients' arises naturally in many situations. Namely, we fix a finite set A of monomials in $x_1, \dots, x_n$ and consider the space $\mathbb{C}^A$ of all linear combinations of these monomials with complex coefficients. The following family of algebraic varieties is naturally related to such data: the parameter space $V_A = \mathbb{C}^A \oplus \dots \oplus \mathbb{C}^A$ is the space of k-tuples $\varphi = (\varphi_1, \dots, \varphi_k)$ of polynomials consisting of these monomials, and the element of the family corresponding to a point φ is the variety $N_\varphi = \{x \mid \varphi_1(x) = \cdots = \varphi_k(x) = 0\}$, that is, the set of solutions of the system of equations $\varphi = 0$. For example, the family of all algebraic hypersurfaces of fixed degree in $\mathbb{C}^n$ is of this type. The invariants of a generic element of such a family are important objects to be studied. For example, according to the fundamental theorem of algebra, if $n = k = 1$ and $A = \{x^0, x^1, \dots, x^d\}$, then a generic element of V_A consists of d points.

Over time it was observed that the algebro-geometric invariants of a generic element of V_A are naturally related to the geometric invariants of the following polytope. Identifying the monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$ with the point $a = (a_1, \dots, a_n)$ in the integer lattice $\mathbb{Z}^n$, we regard the given set A of monomials as a subset of $\mathbb{Z}^n$ and define its Newton polytope as the convex hull $\operatorname{conv} A$. The theory of Newton polytopes studies the invariants of the set of solutions of a generic system of equations $\varphi = 0$ with a prescribed set A of monomials in terms of the invariants of the Newton polytope $\operatorname{conv} A$.

All the invariants mentioned above have been studied from this point of view since the 1970s; see [42], [4], [35], [64], and [40]. We recall these results in § 2.2. Moreover, the resulting connection between algebraic and convex geometry proved to be useful not only for algebraic geometry, but also for convex geometry. For example, it was used to obtain algebro-geometric proofs and generalizations of such fundamental facts in convex geometry as the upper bound conjecture [57], [58] and the Aleksandrov–Fenchel inequality [61], [36].

Since the Newton polytope of a polynomial (that is, the convex hull of its monomials) appears to be an important invariant of this polynomial, we naturally encounter the problem of computing this polytope for various 'universal' polynomials, first and foremost for resultants and discriminants. This problem was solved in [21] and [59]. We recall the solution in § 3.3 and further on.

1.2. Toric varieties. An important tool in the geometric study of objects of any nature (continuous, smooth, analytic, algebraic) is the concept of a variety, that is, the result of gluing together affine charts by means of gluing maps of appropriate regularity (continuous, smooth, analytic, polynomial). In particular, in the theory of Newton polytopes, where the fundamental class of functions is the set of monomials, it is useful to consider varieties whose gluing maps send monomials to monomials, that is, the coordinate representations of the gluing maps are themselves given by monomials. Such varieties are said to be toric. These are exactly those acted on by the complex torus $T = (\mathbb{C} \setminus \{0\})^n = (\mathbb{C}^*)^n$ with a dense orbit. We recall the definition and the structure of toric varieties in § 2.3.

1.3. Tropical geometry. The union of the set of real numbers and an element $-\infty$ is a semifield with respect to the additive operation max and the multiplicative operation $+$. This is called the tropical semifield and is denoted by $\mathbb{T}$. One can develop algebraic geometry over it, called tropical algebraic geometry. This geometry is important because, on the one hand, it deals with purely combinatorial objects (piecewise linear sets and functions) and, on the other hand, its answers to many questions are the same as in 'regular' geometry. Thus, the very presence of such coincidence may give combinatorial answers to complex questions in algebraic geometry. Every such fact is referred to as a 'tropical correspondence theorem'.

The first correspondence theorem which demonstrated the practical importance of tropical algebraic geometry was Mikhalkin's theorem on the coincidence of the Gromov–Witten invariants of the complex plane and the tropical plane. It states that the number of complex algebraic curves of fixed genus and degree passing through a prescribed generic set of points in the complex projective plane is equal to the analogous quantity (appropriately defined) for tropical objects. We recall this in § 3.2.

Proofs of such facts are based on the existence of a tropical analogue (tropicalization) of any algebraic variety. This correspondence is complicated and can be looked at and formalized from various points of view:

– geometric tropicalization, closely related to the concepts of the Bergman cone, universal Gröbner basis, and non-Archimedean amoeba;

– Berkovich analytification of algebraic varieties;

– transition to the fundamental class of a variety in the ring of conditions.

We recall the first construction in §§ 4 and 5, omit the second, and give more details on the third one in §§ 4 and 6 since, as far as we know, the third construction is not as well covered in the literature as the others.

1.4. The ring of conditions. If U and V are subvarieties of complementary dimension in the complex torus $T = (\mathbb{C} \setminus \{0\})^n$, then the shift gU intersects V at the same number of points for almost all $g \in T$, called the intersection number of U and V and denoted by $U \circ V$. Two subvarieties of equal dimension are said to be numerically equivalent if they have the same intersection number with any variety of complementary dimension. We define a k-cycle in T to be a finite formal linear combination of subvarieties of codimension k with integer coefficients. The quotient of the space of k-cycles with respect to numerical equivalence is denoted by C_k. The direct sum $\bigoplus\limits_k C_k$ has the natural structure of a ring with the operation of formal sum of k-cycles and the following intersection-product operation: the product of the classes of subvarieties $U, V \subset T$ is the class of the subvariety $gU \cap V$ with a shift by a generic element $g \in T$.

This definition was first introduced in [11] for other symmetric spaces and then generalized to arbitrary spherical varieties (in particular, arbitrary reductive groups, including the complex torus). It is based on the non-trivial fact that the class of $gU \cap V$ is the same for any generic element $g \in T$. Generally speaking, this is not the case for homogeneous spaces with the action of a non-reductive group. For example, if we replace T by the additive group $\mathbb{C}^3$, then the classes of lines in C_1 are equal only for parallel lines. Hence the intersection $gU \cap V$ of the plane $U = \{y = 0\}$ and the hyperbolic paraboloid $V = \{xy = z\}$ is the line $y = g_2$, $g_2 x = z$, whose class in C_1 depends continuously on the choice of the shift g_2.

In § 7 we demonstrate that an analogue of the ring of conditions can be constructed for a certain natural class of analytic subvarieties of $\mathbb{C}^n$. However, when studying tropical geometry and Newton polytopes, we are mostly interested in the ring of conditions C of the complex torus $T = (\mathbb{C} \setminus \{0\})^n$. For us it plays a role similar to that of the cohomology ring. Knowing the fundamental classes of subvarieties in the cohomology classes of a compact manifold, one can find their intersection numbers in terms of the product operation in this ring. The same holds for the fundamental classes of subvarieties of T in the ring of conditions. Moreover, we show in § 4.3 that the subvarieties of the complex torus have characteristic classes with values in the ring of conditions.

The customary way to develop intersection theory on a topological manifold or an algebraic variety is to consider its cohomology ring or Chow ring. But these rings are too small for many important non-compact manifolds (in particular, homogeneous spaces), and one needs an appropriate compactification to cure this. An important alternative to this approach for homogeneous spaces is to consider their rings of conditions. These rings generalize Schubert calculus to a wide class of non-compact homogeneous spaces. We mainly use the approach based on the ring of conditions of the torus $(\mathbb{C}\setminus\{0\})^n$. The definition of the Chow ring of an algebraic variety will be recalled in § 4.1.5 for completeness.

1.5. Tropicalization of subvarieties of the torus. It turns out that the ring of conditions C admits a tropical correspondence. If we replicate its definition over the tropical semifield $\mathbb{T}$ correctly, the resulting ring $\mathbb{T}C$ of combinatorially-geometric nature (called the ring of tropical fans) turns out to be naturally isomorphic to C.

The ring $\mathbb{T}C$ has a purely combinatorial definition. A k-dimensional tropical fan is a union of k-dimensional polyhedral cones endowed with integer weights which satisfy a balancing condition (see § 4.2.2). Hence the natural isomorphism between C and $\mathbb{T}C$ provides a combinatorial model of the ring of conditions C of the complex torus. (It is important to note that there are still no satisfactory combinatorial models for the rings of conditions of arbitrary reductive groups.)

Thus the fundamental class of a k-dimensional subvariety $V\subset T$ in the ring of conditions C is naturally associated with an element of the ring $\mathbb{T}C$, a k-dimensional tropical fan referred to as the tropicalization of V.

1.6. Tropical compactifications. To calculate the tropical fan of a given k-dimensional variety V it suffices to construct its tropical compactification, that is, a toric compactification such that the closure of V intersects all the orbits properly. (We recall that varieties U and V intersect each other properly if $\operatorname{codim}(U\cap V)=\operatorname{codim}U+\operatorname{codim}V$.) It turns out that the toric variety corresponding to a fan Σ gives a tropical compactification of V if and only if the tropical fan of V is part of the k-dimensional skeleton of Σ.

In particular, each cone Γ in the tropical fan of V corresponds to an orbit O of codimension k in the tropical compactification. The weight of Γ in the tropical fan turns out to be equal to the intersection number of the closure of V with O (which is well defined since the intersections with orbits are proper).

In particular, when V is a hypersurface with equation $\varphi=0$, knowing its tropicalization is equivalent to knowing the Newton polytope N of the polynomial φ since its tropical compactification is given by the toric variety corresponding to N. The tropical fan of the hypersurface is the set of all exterior normals to the edges of N, the cone of all exterior normals to an edge E being endowed with a weight equal to the lattice length of E. In this sense, the concept of tropicalization generalizes that of Newton's polytope from hypersurfaces to arbitrary subvarieties of the complex torus.

1.7. Bergman cone, universal Gröbner basis, non-Archimedean valuations. When an algebraic subvariety V of the complex torus $(\mathbb{C}\setminus\{0\})^n$ is given by a system of equations, there are purely algebraic ways to describe the tropical compactification and the tropical fan of V in terms of the defining system of equations or, more precisely, the ideal I generated by it. These techniques had been known long before the birth of tropical algebraic geometry.

On the one hand, each vector $v = (v_1, \dots, v_n) \in \mathbb{Q}^n$ determines a valuation of the ring of Laurent polynomials in n variables (endowing the variable x_i with weight v_i). The tropical fan of V (as a set, without weights) consists of all valuations $v \in \mathbb{Q}^n$ such that the principal part $\operatorname{in}_v I$ of the ideal I is non-trivial. A related object is the Bergman cone, which had been studied long before the creation of tropical algebraic geometry. We recall that for every valuation there is the so-called Gröbner basis, a finite system of generators f_i of the ideal I whose principal parts $\operatorname{in} f_i$ generate $\operatorname{in}_v I$. Moreover, there is a universal Gröbner basis, a finite system of generators which is a Gröbner basis for all valuations simultaneously. This is why the Bergman cone is indeed a piecewise linear set.

On the other hand, if we extend the field of scalars from $\mathbb{C}$ to the field S of Puiseux series, then to each point $\varphi = (\varphi_1, \dots, \varphi_n) \in (S \setminus \{0\})^n$ of V over the new field (that is, the germ of an analytic curve $\varphi\colon U \to V$, where $U \subset \mathbb{C}$ is a small punctured neighbourhood of the origin) there corresponds a valuation $v \in \mathbb{Q}^n$ whose value at a Laurent polynomial f is equal to the order of zero (or a pole) of $f \circ \varphi$ at the origin. The tropical fan of V (as a set of points, without weights) consists of the valuations of all the points of V over the field of Puiseux series. Therefore, it is also referred to as a non-Archimedean amoeba in view of the obvious analogy with the definition of the classical amoeba.

Both approaches to the algebraic calculation of the tropicalization of a given subvariety will be discussed in detail in § 5.

1.8. Connections with real algebraic geometry. Closely related to the theory of Newton polytopes is Viro's patchworking construction, which has found important applications in questions connected with Hilbert's 16th problem (see, in particular, [65] and [26]). This construction can also be stated in the language of tropical geometry, and the support of the tropical fan of a complex algebraic variety V can be represented as the limit of the set (the amoeba of V)$/t$ as $t \to \infty$ (see, for example, [9]).

The first key applications of tropical algebraic geometry were related specifically to applying the idea of patchworking to new questions in real algebraic geometry, in particular, to the study of Welschinger invariants [47] and Horn's problem on Hermitian matrices [56]. We are not going to cover these issues, which are somewhat outside the main topics of our survey.

2. Newton polytopes and toric varieties

2.1. Mixed volume. Let V be a real n-dimensional vector space with fixed volume form μ. Minkowski [49] introduced the following notions.

Definition 2.1.1. The Minkowski sum of convex bodies $A, B \subset V$ is the convex body

$$A + B = \{a + b \mid a \in A,\ b \in B\}.$$

The mixed volume is a function

$$\mathrm{MV}\colon \{n\text{-tuples of convex bodies in } V\} \to \mathbb{R}$$

which is symmetric and linear with respect to all n arguments and satisfies the equality $\mathrm{MV}_\mu(A,\dots,A) = n!\,\mathrm{Vol}_\mu A$ for every convex $A \subset V$.

When the volume form is clear from the context, we omit it from the notation for the mixed volume. The existence of mixed volume and the assertions below were proved, for example, in [17] and [51]. Uniqueness follows from the following assertion.

Assertion 2.1.2. *The equality*

$$\mathrm{MV}(A_1,\dots,A_n) = \sum_{1\leqslant i_1<\dots<i_k\leqslant n} (-1)^{n-k}\,\mathrm{Vol}(A_{i_1}+\dots+A_{i_k})$$

holds.

Mixed volume is monotone with respect to each argument.

Assertion 2.1.3. *If $B \subset C$, then*

$$\mathrm{MV}(B, A_2,\dots,A_n) \leqslant \mathrm{MV}(C, A_2,\dots,A_n).$$

Example 2.1.4. Let A and B be the horizontal and vertical unit intervals in $\mathbb{R}^2$ with the standard volume form. Then

$$A+B = [0,1]\times[0,1] \quad \text{and} \quad \mathrm{MV}(A,B) = 1.$$

Mixed volume is non-negative, and it is easy to describe all the bodies with mixed volume 0.

Definition 2.1.5. The codimension of an m-tuple of bodies $A_1,\dots,A_m$ is the minimum of $k - \dim(A_{i_1}+\dots+A_{i_k})$ over all subsets $1 \leqslant i_1 < \dots < i_k \leqslant m$.

This definition admits the empty set, which means that the codimension is always non-positive.

Assertion 2.1.6. *The mixed volume of a tuple of bodies is equal to zero if and only if the codimension of the tuple is negative.*

Mixed volume is multiplicative in the following sense.

Assertion 2.1.7. *If $A_1,\dots,A_k$ can be placed in a k-dimensional subspace $U \subset V$ by parallel translations, then*

$$\mathrm{MV}_\mu(A_1,\dots,A_n) = \mathrm{MV}_{\mu'}(A_1,\dots,A_k)\,\mathrm{MV}_{\mu''}(\pi A_{k+1},\dots,\pi A_n),$$

where $\pi\colon V \to V/U$ is the natural projection and $\mu'' = \mu/\mu'$.

Here the mixed volume of $A_1,\dots,A_k$ as subsets of U is well defined since mixed volume is invariant under parallel translations of the components. The most important analytic result concerning mixed volume is the Aleksandrov–Fenchel inequality (see [2]).

Assertion 2.1.8. *The following inequality holds for all A_i, B, and C:*

$$\mathrm{MV}(B,B,A_3,\dots,A_n)\,\mathrm{MV}(C,C,A_3,\dots,A_n)\leqslant \mathrm{MV}(B,C,A_3,\dots,A_n)^2.$$

The classification of cases when this inequality becomes an equality is still an open problem.

An important role in applications to algebraic geometry is played by lattice mixed volume. Let L be a lattice and let $V=L\otimes\mathbb{R}$ be a vector space of dimension n. We write $\operatorname{Vol} B$ for the volume of a measurable set $B\subset V$ with respect to the lattice volume form (such that the volume of the real torus V/L is equal to $n!$). A subspace $U\subset V$ is said to be rational if $\dim U=\dim(U\cap L)$. Speaking of volume in a rational subspace U or in the quotient V/U, we always mean the lattice volume form with respect to the lattice $U\cap L$ or $L/(U\cap L)$, respectively.

A polytope in V is called a lattice polytope if it is the intersection of finitely many half-spaces with rational boundaries and all its vertices lie in L. The mixed volume associated with the lattice volume form is called the lattice mixed volume in V.

Assertion 2.1.9. *The lattice mixed volume of any lattice polytope is an integer.*

This is a consequence, for example, of the following explicit formula for the mixed volume, where $I(A)$ stands for the number of lattice points in a polytope A.

Assertion 2.1.10. *For any bounded lattice polytopes $A_1,\dots,A_n$,*

$$\mathrm{MV}(A_1,\dots,A_n)=\sum_{0\leqslant k\leqslant n,\ 1\leqslant i_1<\dots<i_k\leqslant n}(-1)^{n-k}I(A_{i_1}+\dots+A_{i_k}),$$

where $I(\varnothing)$ is set to be 1.

The following formula is a useful tool for calculating the mixed volume of polytopes.

Definition 2.1.11. The support function of a closed set $A\subset V$ is a continuous function $A(\,\cdot\,)\colon V^*\to\mathbb{R}\cup\{+\infty\}$ given by $A(\gamma)=\sup_{a\in A}\gamma(a)$. The support face for an element $\gamma\in V^*$ is the subset $A^\gamma\subset A$ on which the linear function γ attains its maximum value, that is, $A^\gamma=\{a\in A\mid \gamma(a)=A(\gamma)\}$.

For polytopes the support function is piecewise linear and every face is the support face for some γ. The faces of a finite set A are exactly the intersections of A with the faces of the convex hull of A.

Assertion 2.1.12. *For every bounded convex body A_1 and for all bounded lattice polytopes $A_2,\dots,A_n$,*

$$\mathrm{MV}(A_1,\dots,A_n)=\sum_{\text{primitive }\gamma\in L^*}A_1(\gamma)\,\mathrm{MV}(A_2^\gamma,\dots,A_n^\gamma).$$

We note that the $(n-1)$-dimensional lattice mixed volume on the right-hand side is well defined since the faces $A_2^\gamma,\dots,A_n^\gamma$ can be positioned by parallel translations in the $(n-1)$-dimensional rational subspace $\ker\gamma$, and the sum is well defined since all but finitely many terms are equal to zero (the non-zero terms correspond to those γ which are support functions of the facets of the sum $A_1 + \dots + A_n$). For example, when $A_1 = \dots = A_n = A \ni 0$, this becomes a formula for calculating the volume of A by partitioning A into pyramids with vertices at the origin and with the facets of A as their bases, the volume of each pyramid being one nth of the base volume times the height:

$$\operatorname{Vol} A = \sum_{\text{primitive } \gamma \in L^*} A(\gamma) \operatorname{Vol} A^{\gamma}.$$

Example 2.1.13. The mixed volume of the standard cube $[0,1]^3$ and two copies of the standard simplex in $\mathbb{R}^3$ is obviously equal to 3 by the above formula (the only non-zero term corresponds to $\gamma = (1,1,1)$).

Covolume is also important for connections with algebraic geometry.

Definition 2.1.14 ([14], [15]). Let $C \subset V$ be a convex cone (that is, a convex set with $\mathbb{R}_+ \cdot C = C$). Then the convex bodies $B \subset C$ with bounded $C \setminus B$ form a semigroup P_C with respect to the Minkowski sum. The volume of $C \setminus B$ is called the covolume $\operatorname{coVol} B$. Mixed covolume is the unique symmetric multilinear function $\mathrm{MV}_C\colon \underbrace{P_C \times \dots \times P_C}_{n} \to \mathbb{R}$ such that $\mathrm{MV}_C(B, \dots, B) = n! \operatorname{coVol} B$ for every $B \in P_C$.

Note that this definition is meaningful only when the cone C has full dimension n and is strictly convex (that is, contains no line). A more general definition for an arbitrary C was given in the papers cited in Definition 2.1.14.

The following inversion of the Aleksandrov–Fenchel inequality holds for covolume ([41]; see also [18] in a special case related to hyperbolic geometry).

Theorem 2.1.15. *For any $A_i, B, B' \in P_C$,*

$$\mathrm{MV}_C(B, B, A_3, \dots, A_n) \, \mathrm{MV}_C(B', B', A_3, \dots, A_n) \geqslant \mathrm{MV}_C(B, B', A_3, \dots, A_n)^2.$$

One can calculate mixed covolumes of polytopes by analogy with volumes.

Assertion 2.1.16. *Let $A_1, \dots . A_n \in P_C$ be lattice polytopes. Then*

$$\mathrm{MV}_C(A_1, \dots, A_n) = \sum_{\text{primitive } \gamma \in C^* \cap L^*} A_1(\gamma) \, \mathrm{MV}(A_2^{\gamma}, \dots, A_n^{\gamma}),$$

where $C^* = \bigl\{\gamma \mid \gamma\big|_C > 0\bigr\} \subset V^*$ is the open cone dual to C.

2.2. Newton polytopes. The lattice points $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ will be interpreted as monomials $x^a = x_1^{a_1} \cdots x_n^{a_n}$. Given a finite set $A \subset \mathbb{Z}^n$ of monomials, we consider the space $\mathbb{C}^A = \Bigl\{ \sum_{a \in A} c_a x^a \Bigm| c_a \in \mathbb{C} \Bigr\}$ of their linear combinations. We regard a Laurent polynomial $\varphi \in \mathbb{C}^A$ as a function $\varphi\colon (\mathbb{C} \setminus \{0\})^n \to \mathbb{C}$ (generally speaking, it is undefined on the coordinate planes $x_i = 0$ since a_i may be negative for some $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$). The *restriction* of a polynomial $\varphi(x) = \sum_{a \in A} c_a x^a$ to a subset $B \subset \mathbb{R}^n$ is the polynomial $\varphi(x)|_B = \sum_{a \in A \cap B} c_a x^a$.

Theorem 2.2.1 (Kushnirenko–Bernstein formula [4]). *Let $A_1, \dots, A_n \subset \mathbb{Z}^n$ be finite sets and let $\varphi_i \in \mathbb{C}^{A_i}$ be generic polynomials. Then the number of solutions $x \in (\mathbb{C} \setminus \{0\})^n$ (with multiplicity accounted for) of the system $\varphi_1(x) = \cdots = \varphi_n(x) = 0$ is equal to the lattice mixed volume of the convex hulls of $A_1, \dots, A_n$. The following condition is sufficient for genericity: given any non-zero linear function $\gamma\colon \mathbb{Z}^n \to \mathbb{Z}$, the system $\varphi_1(x)|_{A_1^\gamma} = \cdots = \varphi_n(x)|_{A_n^\gamma} = 0$ has no solutions in $(\mathbb{C} \setminus \{0\})^n$.*

Example 2.2.2. A well-known particular case of this theorem is Bézout's theorem, asserting that generic polynomials of degrees $d_1, \dots, d_n$ in n variables have $d_1 \cdots d_n$ solutions.

Remark 2.2.3. (i) In the statement of Theorem 2.2.1, genericity means that there is a Zariski open subset $U \subset \bigoplus_i \mathbb{C}^{A_i}$ such that the stated equality holds for every element $(\varphi_1, \dots, \varphi_n)$ of U. In particular, the explicit condition for the system of equations stated in the theorem is a generic property.

(ii) Theorem 2.2.1 illustrates the importance of the following notion. The faces of convex polytopes (or convex hulls of finite sets) $A_i \subset \mathbb{R}^n$ are said to be *compatible* if they can be represented as (the convex hulls of) A_i^γ for some linear function $\gamma\colon \mathbb{R}^n \to \mathbb{R}$. Here is an equivalent definition for bounded polytopes. Their faces are said to be compatible if their Minkowski sum is a face of the sum $\sum_i A_i$. (For unbounded polytopes such a reformulation is impossible.)

(iii) The number of solutions with multiplicity can be understood as the dimension of the quotient $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/\langle \varphi_1, \dots, \varphi_n \rangle$ over $\mathbb{C}$ (accordingly, the absence of solutions means that this quotient is trivial). This version of Theorem 2.2.1 holds over any algebraically closed field of characteristic 0.

(iv) It is customary to state Theorem 2.2.1 differently. The convex hull of all the monomials occurring in the polynomial φ_i is called its *Newton polytope*, and the number of common roots in $(\mathbb{C} \setminus \{0\})^n$ of generic polynomials with prescribed Newton polytopes is asserted to be equal to the mixed volume of the Newton polytopes. In this case, the polynomials $\varphi_i(x)|_{A_i^\gamma}$ occurring in the genericity condition are simply the leading non-zero homogeneous components of φ_i in the sense of quasi-degree (when the weight of a monomial x^a is $\gamma(a)$). However, we note that the traditional statement is weaker than the one above. In applications, it is sometimes important to use the theorem with $\varphi_i \in \mathbb{C}^{A_i}$ such that the convex hull of A_i is strictly larger than the Newton polytope (this need not contradict the genericity condition stated in the theorem).

Given any sets $A_1, \dots, A_n \subset \mathbb{R}^n$, we denote the mixed volume of their convex hulls by the formal product $A_1 \cdots A_n$. The value of a power series $f(y) = \sum_{a \in \mathbb{Z}_+^k} c_a y^a$ on a k-tuple of finite sets $A_1, \dots, A_k \subset \mathbb{R}^n$ is defined to be

$$\sum_{a_1 + \cdots + a_k = n} c_a A_1^{a_1} \cdots A_k^{a_k}.$$

Theorem 2.2.4 [35]. *Let $A_1, \dots, A_k \subset \mathbb{Z}^n$ be finite sets and let $\varphi_i \in \mathbb{C}^{A_i}$ be generic polynomials. Then the Euler characteristic of the set of solutions $x \in (\mathbb{C} \setminus \{0\})^n$ of the system $\varphi_1(x) = \dots = \varphi_k(x) = 0$ is equal to $\dfrac{A_1 \cdots A_k}{(1 + A_1) \cdots (1 + A_k)}$. The following condition is sufficient for genericity: given any linear function $\gamma \colon \mathbb{Z}^n \to \mathbb{Z}$, zero is a regular value of the map $(\varphi_1|_{A_1^\gamma}, \dots, \varphi_k|_{A_k^\gamma}) \colon (\mathbb{C} \setminus \{0\})^n \to \mathbb{C}^k$.*

Remark 2.2.5. The expression $\dfrac{A_1 \cdots A_k}{(1 + A_1) \cdots (1 + A_k)}$, which provides the answer, is the value at $A_1, \dots, A_k$ of the power series expansion of the rational function $\dfrac{s_1 \cdots s_k}{(1 + s_1) \cdots (1 + s_k)}$ at the origin. For zero-dimensional complete intersections ($k = n$), the result becomes the Kushnirenko–Bernstein formula without multiple roots. For curves ($k = n-1$) the Euler characteristic is equal to $-\operatorname{MV}(A_1, \dots, A_{n-1}, A_1 + \dots + A_{n-1})$. For hypersurfaces ($k = 1$), it is equal to $(-1)^{n-1} n! \operatorname{Vol} A_1$.

A similar fact is also known for germs of analytic functions. Given any $I \subset \{1, \dots, n\}$, we denote the coordinate subspace $\{x_i = 0 \text{ for } i \notin I\}$ by $\mathbb{R}^I \subset \mathbb{R}^n$, the positive octant $\{x_i \geqslant 0 \text{ for } i \in I \text{ and } x_i = 0 \text{ for } i \notin I\}$ by $\mathbb{R}^I_+$, the intersection $B \cap \mathbb{R}^I$ with any set $B \subset \mathbb{R}^n$ by B^I, and the $|I|$-dimensional lattice mixed covolume $\operatorname{MV}_{\mathbb{R}^I_+}(B^I_1, \dots, B^I_{|I|})$ by $B^I_1 \cdots B^I_{|I|}$.

Definition 2.2.6. The Newton polytope $\mathscr{N}_f$ of a germ $f \in \mathbb{C}[[x_1, \dots, x_n]]$ is the convex hull of the union of $a + \mathbb{R}^n_+$ over all $a \in \mathbb{Z}^n$ such that the monomial x^a occurs with a non-zero coefficient in f. We say that $\mathscr{N}_f$ is convenient if it has a common point with every coordinate axis. The Newton diagram Δ_f is the union of all bounded faces of the Newton polytope $\mathscr{N}_f$. The restriction $f\big|_{\Delta_f}$ is called the principal part of f. Given any linear function $\gamma \in (\mathbb{R}^n_+)^*$, we write f^γ for the first non-zero homogeneous (in the sense of the quasi-degree $\deg x^a = \gamma(a)$) component of the series f (note that it depends only on the principal part of f).

Theorem 2.2.7. *Suppose that the germs of analytic functions*

$$(f_1, \dots, f_k) \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$$

have convenient Newton polytopes $A_1, \dots, A_k$ and their principal parts satisfy the following genericity condition: zero is a regular value of the polynomial map $(f_1^\gamma, \dots, f_k^\gamma) \colon (\mathbb{C} \setminus \{0\})^n \to \mathbb{C}^k$ for every $\gamma \in (\mathbb{R}^n_+)^*$. Then the equations $f_1 = \dots = f_k = 0$ determine an isolated singularity of the complete intersection at the origin, the Milnor number of which is equal to $(-1)^{n-k} \displaystyle\sum_{I \subset \{1,\dots,n\}} \frac{A_1^I \cdots A_k^I}{(1 + A_1^I) \cdots (1 + A_k^I)}$, where the term with $I = \varnothing$ is equal to 1 by definition.

An equivalent formula was obtained in [1] in the case $k = n$ without using the notion of mixed covolume.

Theorem 2.2.7 generalizes to the monodromy ζ-function of the germ f_1 on the variety $f_2 = \dots = f_k = 0$. To state this generalization, we define the following expression for polytopes $A_1, \dots, A_k \in P_{\mathbb{R}^k_+}$:

$$t^{A_1} \cdot A_2 \cdots A_k = \sum_{\text{primitive } \gamma \in \mathbb{Z}^n_+} \operatorname{MV}(A_2^\gamma, \dots, A_n^\gamma) \log\bigl(1 - t^{A_1(\gamma)}\bigr).$$

By Assertion 2.1.16, the exponent of this expression is a polynomial of degree $A_1 \cdots A_k$. The notation $t^{A_1} \cdot \varphi(A_2, \dots, A_k)$, where φ is an arbitrary power series, has similar meaning.

Theorem 2.2.8 ([64] for $k = 1$ and [50]). *Under the hypotheses of the previous theorem, the logarithm of the characteristic polynomial of the cohomology monodromy of the Milnor fibration of f_1 on the complete intersection $f_2 = \cdots = f_k = 0$ is equal to*

$$(-1)^{n-k} \sum_{I \subset \{1,\dots,n\}} \frac{t^{A_1^I} \cdot A_2^I \cdots A_k^I}{(1 + A_1^I) \cdots (1 + A_k^I)},$$

where the term with $I = \varnothing$ is equal to $\log(1 - t)$ *by definition.*

2.3. Toric varieties and polytopes.

2.3.1. *Motivational example.* It was discovered in [34] that toric varieties help in proving the assertions of § 2.2 because they give smooth compactifications and resolutions of generic objects studied therein.

We give a model example. Let $f = \sum_{a \in N} c_a x^a$ be a generic Laurent polynomial in two variables with Newton polytope $N \subset \mathbb{Q}^2$. We wish to construct a smooth compactification of the curve $C = \{f = 0\} \subset (\mathbb{C} \setminus \{0\})^2$. The simplest compactification (the closure of C in $(\mathbb{C} \setminus \{0\})^2 \subset \mathbb{CP}^2$) appears to be smooth only for a few polytopes N.

To construct a smooth compactification, we write $\mathbb{CP}^N$ for the projective space whose standard coordinates y_a are labelled by the lattice points $a \in N$ and consider the embedding $j \colon (\mathbb{C} \setminus \{0\})^2 \to \mathbb{CP}^N$ sending any $x \in (\mathbb{C} \setminus \{0\})^2$ to the point with homogeneous coordinates $y_a = x^a = x_1^{a_1} x_2^{a_2}$. It turns out that if the coefficients of f are generic (in the sense of Remark 2.2.3), then the closure of the image of C under j is smooth.

This compactification of C is said to be toric, and the closure of the image $j\bigl((\mathbb{C} \setminus \{0\})^2\bigr) \subset \mathbb{CP}^N$ of the complex torus is called the toric variety X_N. Identifying the torus $(\mathbb{C} \setminus \{0\})^2$ with its image $j\bigl((\mathbb{C} \setminus \{0\})^2\bigr) \subset X_N$, we can regard X_N as a compactification of the torus $(\mathbb{C} \setminus \{0\})^2$. Then we can forget about the ambient space $\mathbb{CP}^N$ and define the toric compactification of C as the closure of C in the toric variety $X_N \supset (\mathbb{C} \setminus \{0\})^2 \supset C$.

Some of the above assertions need a slight correction when we pass from the two-dimensional case to many dimensions. In particular, there are three-dimensional polytopes N such that j is not an embedding or the closure of $j(C)$ is not smooth. This correction follows naturally from the information about toric varieties that will be briefly recalled below.

2.3.2. *The notion of toric variety.* A toric variety is an (irreducible) n-dimensional algebraic variety X acted on by the complex torus $T \simeq (\mathbb{C} \setminus \{0\})^n$ in such a way that one of the orbits is (Zariski) dense in X. In this definition X is often assumed to be normal, but we do not make this assumption.

Toric varieties (X_1, T_1) and (X_2, T_2) are said to be isomorphic if there is a pair of isomorphisms $X_1 \to X_2$ and $T_1 \to T_2$ compatible with the actions of T_i on X_i.

Toric varieties (X_1, T) and (X_2, T) are said to be canonically isomorphic if there is an isomorphism $X_1 \to X_2$ compatible with the actions of T on X_i.

In what follows we explicitly construct a projective toric variety X_A corresponding to an arbitrary finite set $A \subset \mathbb{Z}^n$. We also construct a normal toric variety X_Σ corresponding to an arbitrary fan Σ (a fan is a piecewise linear combinatorial object to be defined below).

These constructions are important because of their universality: every projective (resp. normal) toric variety is isomorphic to X_A for an appropriate A (resp. isomorphic to X_Σ for an appropriate Σ).

Remark 2.3.1. It is important to separate the problem of describing the isomorphism classes of toric varieties (with a full answer given below) from the much more delicate problem of describing the group of abstract automorphisms of a toric variety and its orbits (by an abstract automorphism we mean an automorphism of the algebraic variety that is not necessarily compatible with the torus action) as well as from the open problems on diffeomorphisms of toric varieties (in particular, on their cohomological rigidity). These extensively developing areas of research require separate surveys and are not covered here.

2.3.3. *The projective toric variety X_A.* Given a finite set A, we write $\mathbb{CP}^A$ for the standard $(|A|-1)$-dimensional projective space whose homogeneous coordinates y_a are labelled by the elements $a \in A$. To every subset $B \subset A$ there corresponds a coordinate subspace $\mathbb{CP}^B = \{y_a = 0 \text{ for } a \notin B\} \subset \mathbb{CP}^A$ containing the standard complex torus $(\mathbb{C} \setminus \{0\})^B = \{y_a \neq 0 \text{ for } a \in B \text{ and } y_a = 0 \text{ for } a \notin B\}$.

Let L be the lattice of characters of the n-dimensional complex torus T. If we fix a coordinate system $T \simeq (\mathbb{C}\setminus\{0\})^n$ and the corresponding coordinates $L \simeq \mathbb{Z}^n$, then the value of a character $a \in L$ at a point $x \in T$ is the monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$. Hence we can also denote this value by x^a regardless of coordinates.

Definition 2.3.2. The toric variety X_A corresponding to a finite subset $A \subset L$ is the closure in $\mathbb{CP}^A$ of the image of the homomorphism $j_A \colon T \to (\mathbb{C} \setminus \{0\})^A$ sending any $x \in T$ to the point with homogeneous coordinates $y_a = x^a$, $a \in A$. The action of the torus T on X_A is defined to be the composite of the homomorphism $j_A \colon T \to (\mathbb{C} \setminus \{0\})^A$ and the standard action of $(\mathbb{C} \setminus \{0\})^A \subset \mathbb{CP}^A$ on $\mathbb{CP}^A$ by coordinate-wise product.

The kernel $\ker_A$ of j_A, equal to $\{x \mid x^a = x^b \text{ for all } a, b \in A\}$ by construction, can be non-trivial. This happens if and only if A can be positioned in a proper sublattice of L by a parallel translation. In particular, the dense orbit in X_A, which is equal to $j_A(T) = X_A \cap (\mathbb{C} \setminus \{0\})^A$, can be identified with the quotient $T/\ker_A$. Hence the dimension of X_A is equal to that of the convex hull of A.

However, we note that the toric variety X_A does not change under parallel translations of A (that is, $X_A = X_{A+a}$ for every $a \in L$) or under changes of the ambient lattice (that is, $X_A = X_B$ if $B \subset L'$ is the image of A under an embedding of lattices $L \subset L'$). Therefore, every projective toric variety can be represented in the form X_A, where A cannot be shifted to a proper sublattice of L by a parallel translation.

Remark 2.3.3. (i) In view of the aforesaid, we can and will always assume without loss of generality that A cannot be shifted to a proper sublattice of L by a parallel translation. In this case $\ker_A$ is trivial, $\dim X_A = \dim L$, and T can be identified with a dense orbit in X_A.

(ii) The set of lattice points of a lattice polytope P can often be taken as A. However, it is important to remember that in dimension $n \geqslant 3$ there are polytopes $P \subset \mathbb{Z}^n$ whose set of lattice points $P \cap \mathbb{Z}^n$ does not satisfy the assumption in (i). The simplest example is the Reeve tetrahedron, which contains no lattice points except for the vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,k)$, $k > 1$.

(iii) A polytope whose set of lattice points cannot be moved to a proper sublattice by a parallel translation is said to be a spanning or reduced polytope. In accordance with (i), we consider only reduced polytopes in what follows.

2.3.4. *Limit points of curves in X_A.* Since X_A is compact, $\lim_{t\to 0} j_A(\varphi(t))$ is contained in X_A for every analytic curve germ $\varphi\colon (\mathbb{C}\setminus\{0\}) \to T$. Conversely, every point of X_A can be represented as such a limit. This is similar to the representation of infinitely remote points in a projective space by the pencils of parallel lines passing through them. To complete this useful analogy, we find the coordinate tori $(\mathbb{C}\setminus\{0\})^B$ of the projective space $\mathbb{CP}^A = \bigsqcup_{B\subset A} (\mathbb{C}\setminus\{0\})^B$ which contain the limit.

Let L^* be the lattice dual to the lattice of characters of T. It is called the lattice of one-parameter subgroups. Indeed, for every point l of L^* there is a one-parameter subgroup $\psi\colon \mathbb{C}\setminus\{0\} \to T$ uniquely determined by $\psi(t)^a = t^{l(a)}$ for every $a \in L$. If we choose coordinates $T \simeq (\mathbb{C}\setminus\{0\})^n$ and $L^* \simeq \mathbb{Z}^n$, then $\psi(t) = t^l = (t^{l_1},\dots,t^{l_n})$ is a Veronese curve. Thus, we denote this one-parameter subgroup by t^l regardless of coordinates. When speaking of the one-parameter subgroup t^l considered only for small values of t, we use the term *semi-one-parameter*.

Using this notation, any analytic curve germ $\varphi\colon \mathbb{C}\setminus\{0\} \to T$ can be written as

$$\varphi(t) = \frac{\tilde\varphi(t)}{t^l}\,,$$

where $\tilde\varphi(0) \in T$ is well defined, and the division is understood in the sense of the group operation on T. Indeed, if we choose coordinates $T \simeq (\mathbb{C}\setminus\{0\})^n$ and $L^* \simeq \mathbb{Z}^n$, then the coordinates l_i of the covector l should be equal to the degrees of the leading monomials in the Laurent expansions of the components $\varphi_i(t)$ of the curve $\varphi(t)$, that is, $l = \deg\varphi$. Thus, we denote l by $\deg\varphi$ regardless of coordinates.

Using the notation in Definition 2.1.11 for the support face A^l and the support function $A(l)$, we can write the homogeneous coordinates y_a of the limit point

$$\lim_{t\to 0} j_A(\varphi(t)) = \lim_{t\to 0} j_A\biggl(\frac{\tilde\varphi(t)}{t^l}\biggr)$$

in the following form after multiplying by $t^{A(l)}$ at the same time:

$$y_a = \lim_{t\to 0} t^{A(l)-l(a)}\tilde\varphi(t)^a = \begin{cases} \tilde\varphi(0)^a & \text{if } l(a) = A(l), \\ 0 & \text{if } l(a) < A(l). \end{cases} \tag{2.1}$$

Corollary 1. The limit point of every analytic curve germ $\varphi\colon \mathbb{C}\setminus\{0\} \to T$ belongs to the coordinate torus $(\mathbb{C}\setminus\{0\})^{A^{\deg\varphi}} \subset \mathbb{CP}^A$. Moreover, since it follows from the valuation criterion for completeness that every point of X_A can be represented as the limit of an analytic curve germ $\varphi\colon \mathbb{C}\setminus\{0\}\to T$, the coordinate tori $(\mathbb{C}\setminus\{0\})^B$ for other subsets $B\subset A$ contain no point of X_A.

Corollary 2. Every point X_A can be written as the limit of a shifted semi-one-parameter group c/t^l, $c\in T$, $l\in L^*$. Two shifted semi-one-parameter groups c/t^l and $c'/t^{l'}$ tend to the same point if and only if

$$A^l = A^{l'} \quad \text{and} \quad \frac{c}{c'}\in \ker_{A^l}.$$

This enables us to represent points in X_A as classes of shifted semi-one-parameter groups with respect to the equivalence relation above. This is similar to representing points in a projective space by pencils of lines.

2.3.5. *The partition of X_A into orbits.* By construction, orbits in X_A are intersections of X_A with orbits of the standard action of $(\mathbb{C}\setminus\{0\})^A$ on $\mathbb{CP}^A$, that is, with complex tori $(\mathbb{C}\setminus\{0\})^B$. However, the above corollaries which follow from (2.1) show that such an intersection $(\mathbb{C}\setminus\{0\})^B\cap X_A$ is non-empty only when B is a *face* of A, that is, the intersection of A with a face of its convex hull. Thus, the variety X_A is subdivided into orbits $T_B=(\mathbb{C}\setminus\{0\})^B\cap X_A=j_B(T)\simeq T/\ker_B$ over all faces $B\subset A$, and we have $\dim T_B=\dim B$. Note that this correspondence between the faces of A and the orbits of X_A preserves adjacency. Moreover, the closure of T_B in X_A can naturally be identified with the toric variety X_B.

2.3.6. *Covering X_A by charts.* We write $M_a\simeq\mathbb{C}^{|A|-1}$ for the affine chart of $\mathbb{CP}^A$ given by the condition $y_a\neq 0$. The variety X_A is covered by the charts $X_A\cap M_a$, $a\in A$, which are affine varieties by construction. We denote them by X_{A_a}.

The chart X_{A_a} covers all the orbits T_B, where $B\subset A$ is a face containing a. In particular, X_A is fully covered by the charts X_{A_a} corresponding to vertices $a\in A$ (that is, vertices of the convex hull of A).

The chart M_a is endowed with standard coordinates $z_b=y_b/y_a$, $b\in A$, $b\neq a$. In these coordinates, the affine variety X_{A_a} can be described as the closure of the image of the map $j_{A,a}\colon T\to M_a$ sending every $x\in T$ to the point with coordinates $z_b=x^{b-a}$. In particular, the restrictions of the polynomials in z_b to X_{A_a} produce Laurent polynomials of the form $\sum\limits_{a'\in A_a} c_{a'}x^{a'}$, where $A_a\in L$ is the semigroup generated by the differences $b-a$, $b\in A$. (Here and in what follows we adopt the convention that all semigroups contain zero.)

In other words, the ring of regular functions on the affine variety X_{A_a} depends only on the semigroup A_a, rather than on the whole of A, and it is its semigroup algebra $\mathbb{C}[A_a]$. We recall that an affine variety is completely determined by its ring of regular functions: $X_{A_a}=\operatorname{Spec}\mathbb{C}[A_a]$. In particular, if A_a and B_b are isomorphic semigroups, then so are the affine toric varieties X_{A_a} and X_{B_b}. This justifies the notation X_{A_a} and the following definition.

Definition 2.3.4. The affine toric variety X_S corresponding to a finitely generated semigroup $S\subset L$ with zero is the variety $\operatorname{Spec}\mathbb{C}[S]$ with an action of the torus $T=\operatorname{Spec}\mathbb{C}[L]$ induced by the embedding $\mathbb{C}[S]\subset\mathbb{C}[L]$.

To put it simply, X_S is the closure of the image of the map $j_B\colon T\to\mathbb{C}^B$ sending every $x\in T$ to the point with coordinates $z_b=x^b$, $b\in B$, where B is an arbitrary finite system of generators of S. (Independence of the choice of generators follows from the coincidence with the first definition. The torus action on X_S is defined to be the composite of the embedding $j_B\colon T \to (\mathbb{C}\setminus\{0\})^B$ and the standard coordinate-wise product action of $(\mathbb{C}\setminus\{0\})^B$ on $\mathbb{C}^B$.)

Remark 2.3.5. By analogy with Remark 2.3.3, we can and will assume without loss of generality that the semigroup S generates the ambient lattice $\mathbb{Z}^n$. Speaking of a semigroup, we always mean that it is finitely generated and contains zero.

In particular, this is the case for the semigroups A_a, $a \in A$, in view of Remark 2.3.3.

Example 2.3.6. Note that the affine toric variety X_S is isomorphic to $\mathbb{C}^n = \operatorname{Spec}\mathbb{C}[x_1,\dots,x_n]$ if and only if the semigroup S is free, that is, isomorphic to $\{(a_1,\dots,a_n) \in \mathbb{Z}^n \mid a_i \geqslant 0\}$.

Suppose that $A \subset \mathbb{Z}^n$ and $a \in A$. The semigroup A_a is free if and only if

1) a is a vertex of the convex hull P of A;

2) the lattice points a_i closest to a on the adjacent edges of P are contained in A;

3) the vectors $a_i - a$ form a basis of the lattice.

Remark 2.3.7. Multiplicity is a measure of non-smoothness of an algebraic variety at a given point. By a formula of Kushnirenko, the multiplicity of the affine toric variety X_S at 0 is equal to the lattice volume of the difference of the convex hulls of S and $S\setminus\{0\}$.

Example 2.3.8. We can easily conclude from definitions that the semigroup ring $\mathbb{C}[S]$ is integrally closed if and only if the semigroup $S \subset \mathbb{Z}^n$ contains all the lattice points of its convex hull. We call such a semigroup integrally closed and conclude that the affine toric variety $X_S = \operatorname{Spec}\mathbb{C}[S]$ is normal if and only if the semigroup S is integrally closed.

2.3.7. *Smoothness and normality of the projective toric variety X_A.* According to the example above, if the semigroups A_a are free for all vertices $a \in A$, then the variety X_A is covered by smooth charts X_{A_a}. This proves the 'if' part of the following assertion.

Assertion 2.3.9. *The variety X_A is smooth if and only if the semigroups A_a are free for all vertices $a \in A$.*

The 'only if' part is not so important for us. It can also be deduced directly from the definitions. This assertion motivates the following definitions.

Definition 2.3.10. 1) An n-dimensional polytope is said to be simple if each of its vertices is incident to n edges.

2) A simple n-dimensional lattice polytope is said to be integrally simple or (mostly in symplectic contexts) is called a Delzant polytope if, for every vertex, the vectors issuing from it towards the closest lattice points on incident edges form a basis of the lattice $\mathbb{Z}^n$.

3) A finite set $A \subset \mathbb{Z}^n$ is said to be integrally simple if its convex hull is integrally simple and the lattice points inside each of its edges that are closest to the endpoints lie in A.

Remark 2.3.11. (i) By Example 2.3.6, the last assertion can be restated as follows: the variety X_A is smooth if and only if the set $A \subset \mathbb{Z}^n$ is integrally simple.

(ii) We recall that, in accordance with Remark 2.3.3, we always assume that A cannot be moved to a proper sublattice by a parallel translation. The assertion in part (i) does not hold without this assumption. For example, if A is the Reeve tetrahedron (see Remark 2.3.3), then A is not integrally simple, but X_A is smooth (this is merely a projective space).

Similar to smoothness, a criterion for the normality of a toric variety can be obtained by passing to affine charts.

Definition 2.3.12. A set $A \subset \mathbb{Z}^n$ is said to be integrally closed if for all vertices (and hence for all points) $a \in A$ the semigroups A_a are integrally closed (see Example 2.3.8). A lattice polytope is said to be integrally closed if the set of its lattice points is integrally closed.

Assertion 2.3.13. *The variety X_A is normal if and only if A is integrally closed.*

2.4. Toric varieties and fans.

2.4.1. *Isomorphism of toric varieties and fans.* The covering of a toric variety by charts shows that the equality $A = B$ is not required for the normal varieties X_A and X_B to be isomorphic. One needs a much weaker condition, namely the equality of their dual fans in the following sense.

By a *cone* in $\mathbb{Q}^n$ we mean the set of linear combinations of a given finite set of vectors with positive coefficients. Note that it is more customary to define cones as closed sets (that is, sets of linear combinations with non-negative coefficients), but in this subsection it is convenient for us to understand cones as relatively open sets (that is, open in the topology of their linear span).

By a *face* we mean the intersection of the closure of the cone with the boundary of a half-space containing it. A cone is said to be *strictly convex* (or *pointed*) if its minimal face is a point (that is, the cone contains no lines). A *fan* is a set of strictly convex cones such that, along with any cone, it contains all of its faces and the intersection of the closures of any two cones is a face shared by them. A fan is said to be *complete* if the ambient space is the union of its cones.

Definition 2.4.1. Given a polytope or a finite set A in a $\mathbb{Q}$-vector space V, we define the dual cone of a face $B \subset A$ to be the set of exterior normals to B, that is, linear functions $l \in V^*$ whose restrictions to A attain their maxima precisely at the points in B (in other words, points l such that $B = A^l$ is a support face). The dual fan of A is the set of dual cones of all the faces of A.

If the dual fan of A' is a subdivision of the dual fan of A and the toric varieties $X_{A'}$ and X_A are normal, then the identification of their dense orbits (canonically isomorphic to the torus T) extends to a map $\pi_{A',A}\colon X_{A'} \to X_A$.

In fact, for every face $B' \subset A'$ there is a unique face $B \subset A$ whose dual cone contains the dual cone of B'. Since X_A is normal, we have $\ker_{B'} \subset \ker_B$ by Assertion 2.3.13. Therefore, we can define $\pi_{A',A}$ on the orbit $T_{B'} = T/\ker_{B'}$ to be the factorization map to $T_B = T/\ker_B$.

It follows from the existence of the maps $\pi_{A',A}$ and $\pi_{A,A'}$ that normal toric varieties X_A and $X_{A'}$ are isomorphic if the dual fans of A and A' are isomorphic (that is, can be mapped to one another by an isomorphism between the ambient spaces). This implies that a toric variety X_A can be recovered from the dual fan Σ_A of A even without knowing A itself.

Our aim is to describe a construction of this recovery. First we give a general definition and then explain it using a more elementary and explicit coordinate language in the most important case of smooth varieties.

2.4.2. *The toric variety corresponding to a fan.*

Definition 2.4.2. Let V be a space endowed with a lattice L and let Σ be a fan in V^*. For every cone $C \in \Sigma$ we define the dual semigroup $C^\times$ as the intersection of the dual cone with L. An adjacency $C' \subset C$ gives an embedding of group algebras $j_{C',C}\colon \mathbb{Z}[C'] \subset \mathbb{Z}[C]$. Gluing the affine toric varieties $X_{C^\times} = \operatorname{Spec} \mathbb{Z}[C^\times]$, $C \in \Sigma$, along the maps induced by the embeddings $j_{C',C}$, we obtain a variety denoted by X_Σ. Since the gluing maps are compatible with the actions of the torus T on the affine charts, the torus action is well defined on the whole of X_Σ. We call it the toric variety corresponding to the fan Σ.

Remark 2.4.3. (i) When the fan Σ is generated by a single full-dimensional cone C (that is, consists of all its faces), the toric variety X_Σ coincides with the affine variety $X_{C^\times}$.

(ii) When the maximal cones of the fan are generated by bases of the lattice (and only in this case), the variety X_Σ turns out to be smooth. Then the definition becomes a construction of X_Σ by gluing together several charts (isomorphic to $\mathbb{C}^n$) by means of the monomial gluing maps encoded in the fan. In what follows (see § 2.4.3) we give an explicit description of this construction in coordinates.

(iii) When $\Sigma = \Sigma_A$ is the dual fan of an integrally closed subset A in the lattice of characters of T, we have repeated *verbatim* the construction of a covering of the normal toric variety X_A by affine charts.

Even when A is not integrally closed, the embedding of the torus T in the toric varieties X_A and X_{Σ_A} extends to a map $X_{\Sigma_A} \to X_A$, which is a normalization of X_A.

Even when X_A is not normal, the normalization map is often one-to-one. This occurs if and only if the differences of the points of any face $B \subset A$ generate a saturated sublattice in L.

(iv) For arbitrary fans this construction establishes a one-to-one correspondence between normal toric varieties (that is, ones possessing a torus action with dense orbit) and rational fans. This correspondence sends compact varieties to complete fans, projective varieties to convex or coherent fans (that is, the dual fans of polytopes), and smooth varieties to smooth or non-singular fans (that is, fans whose full-dimensional cones are generated by lattice bases). Convex fans are often referred to as normal, but we shall not use this term. (This property of a fan is unrelated to the normality of the corresponding toric variety, which can lead to ambiguities in the context of algebraic geometry.)

(v) Not all fans are convex. The simplest example is given by the complete fan whose two-dimensional cones are generated by edges in the set

$$\{\text{the complete graph on the vertices } A_1, B_1, C_1, A_2, B_2, C_2\} \setminus \{A_1B_2, B_1C_2, C_1A_2\},$$

where $A_1B_1C_1A_2B_2C_2$ is a triangular prism with centre $0 \in \mathbb{R}^3$. But if a fan is convex, it possesses a dual polytope whose set of lattice points is integrally closed. For example, for any polytope P, the set of lattice points of its homothety $\widetilde{P} = (\dim P + 1) \cdot P$ is integrally closed. This means, in particular, that the normal toric variety X_Σ corresponding to a convex fan Σ can be realized as the projective toric variety X_A corresponding to an integrally closed set A of lattice points.

(vi) A convex fan is non-singular if and only if all (or, equivalently, some) of its dual polytopes are integrally simple. This means, in particular, that the smooth toric variety X_Σ corresponding to a smooth convex fan Σ can be realized as the projective toric variety X_A corresponding to the set A of lattice points of an integrally simple polytope.

(vii) By construction, the orbits of X_Σ are in a one-to-one correspondence with the cones in Σ. The orbit corresponding to a cone $C \in \Sigma$ is denoted by T_C. It is naturally isomorphic to the quotient $T/\{t^l,\ l \in C\}$.

(viii) If a fan Σ' is a subdivision of Σ, then there is a natural map $\pi_{\Sigma',\Sigma}\colon X_{\Sigma'} \to X_\Sigma$. For every cone $C' \in \Sigma'$ lying in a cone $C \in \Sigma$, this map sends the orbit $T_{C'} = T/\{t^l,\ l \in C'\}$ to the orbit $T_C = T/\{t^l,\ l \in C\}$ by taking the quotient relative to $\{t^l,\ l \in C\}$.

2.4.3. *Gluing a smooth toric variety in accordance with a fan.* In the case when every maximal cone C of a fan Σ is generated by a lattice basis v_C (and only in this case), the variety X_Σ is smooth. Then the definition of X_Σ takes the form of constructing it by gluing together several charts (isomorphic to $\mathbb{C}^n$ and labelled by the maximal cones C) by means of monomial gluing maps encoded in the transition matrices between the bases v_C. We now describe this construction explicitly.

The bases $v = (v_1, \dots, v_n)$ of the lattice L^* of one-parameter subgroups of the torus T are in a one-to-one correspondence with the monomial parameterizations $h_v\colon (\mathbb{C}\setminus\{0\})^n \to T$ of this torus. Namely, $h_v(t_1, \dots, t_n) = t_1^{v_1} \cdots t_n^{v_n}$, where $t = (t_1, \dots, t_n)$ are the standard coordinates on $(\mathbb{C}\setminus\{0\})^n$. In particular, any two bases v and $\widetilde{v}$ identify T with two copies of the standard torus $(\mathbb{C}\setminus\{0\})^n$. We denote the coordinates on these copies by t and $\widetilde{t}$, respectively.

The coordinates $\widetilde{t}$ can be expressed in terms of t using the transition matrix, $v^\top = \mathcal{C} \cdot \widetilde{v}^\top$. This is a monomial expression. Let c^i (resp. c_i) be the rows (resp. columns) of the matrix $\mathcal{C}$. Then we have an equality

$$\widetilde{t}_1^{\widetilde{v}_1} \cdots \widetilde{t}_n^{\widetilde{v}_n} = t_1^{v_1} \cdots t_n^{v_n} = t_1^{c^1 \cdot \widetilde{v}^\top} \cdots t_n^{c^n \cdot \widetilde{v}^\top} = (t^{c_1})^{\widetilde{v}_1} \cdots (t^{c_n})^{\widetilde{v}_n}$$

for points in T, that is, in coordinates,

$$\widetilde{t}_i = t^{c_i}. \tag{2.2}$$

This map $(\mathbb{C}\setminus\{0\})^n_t \to (\mathbb{C}\setminus\{0\})^n_{\widetilde{t}}$ has the following important properties:

(a) *Extendability.* If the first k vectors of a basis v lie in the cone generated by the basis $\widetilde{v}$, then the map $g_{v,\widetilde{v}}$ given by (2.2) is well defined on $\mathbb{C}^k \times (\mathbb{C}\setminus\{0\})^{n-k}$ (and not only on the subset $(\mathbb{C}\setminus\{0\})^n$). This is because the first k rows of the transition matrix $\mathcal{C}$ consist of non-negative numbers.

(b) *Invertibility.* If, moreover, the first k vectors of v coincide with those of $\widetilde{v}$, then $g_{v,\widetilde{v}}$ is a one-to-one correspondence $\mathbb{C}^k \times (\mathbb{C} \setminus \{0\})^{n-k} \to \mathbb{C}^k \times (\mathbb{C} \setminus \{0\})^{n-k}$. Indeed, $g_{\widetilde{v},v}$ is the well-defined inverse map.

(c) *Separability.* If, moreover, the intersection of the cones generated by v and $\widetilde{v}$ is their common face generated by $v_1, \dots, v_k$, then the result of gluing two copies of $\mathbb{C}^n \supset \mathbb{C}^k \times (\mathbb{C} \setminus \{0\})^{n-k}$ by means of the map $g_{v,\widetilde{v}}$ has a separable topology. Indeed, if two points not identified by $g_{v,\widetilde{v}}$ could not be separated, then, according to the valuative criterion for completeness, they would be limits of the germ of the same curve $\gamma\colon \mathbb{C} \to T$ (more precisely, of its pull-backs under the parameterizations $h_v, h_{\widetilde{v}}\colon (\mathbb{C} \setminus \{0\})^n \to T$). Hence the one-parameter group $\deg\gamma$ that approximates the curve $\gamma \in L^*$ asymptotically (see p. 104) would lie in the cones generated by both these bases but not in their common face. This contradicts our assumptions.

The properties of the maps $g_{v,\widetilde{v}}$ enable us to use them to glue together smooth charts to obtain the toric variety corresponding to a non-singular fan Σ in L^* (that is, a fan each of whose full-dimensional cones C is generated by a lattice basis v_C). Namely, we take the chart $M_C = \mathbb{C}^n$ for every such cone C and glue these charts together by means of the gluing maps $g_{v_C,v_{\widetilde{C}}}\colon M_C \to M_{\widetilde{C}}$. The action of the torus T on each chart extends to an action on the whole resulting variety X_Σ.

The resulting action has a dense orbit, which can be identified with T. The parameterizations $h_{v_C}\colon (\mathbb{C}\setminus\{0\})^n \to T$ introduced above can be extended to become the charts $M_C \to X_\Sigma$ of an atlas.

All smooth toric varieties can be obtained by means of this construction.

2.5. Toric resolutions and compactifications. Consider an algebraic subset $V \subset T$ of the complex torus, that is, a subset given by polynomial equations $\varphi_1 = \dots = \varphi_k = 0$. Motivated by § 2.3.1, we want to find a toric variety $X_\Sigma \supset T$ such that the closure of V in it is smooth.

It turns out that if the maps φ_i in the equations of V are generic Laurent polynomials in the spaces $\mathbb{C}^{A_i}$, then the toric variety X_Σ has the desired property for any non-singular fan Σ which is a subdivision of the dual fans of $A_1, \dots, A_k$. More precisely, the following theorem holds.

Theorem 2.5.1. 1) *For every k-tuple of finite subsets $A_1, \dots, A_k$ in the lattice L there is a complete non-singular fan subdividing $\Sigma_{A_1}, \dots, \Sigma_{A_k}$ (that is, a fan such that every cone in it is generated by a part of a basis of the lattice L^* and lies in some cone of each fan Σ_{A_i}).*

2) *The k-tuples of Laurent polynomials $(\varphi_1, \dots, \varphi_k)$ satisfying the condition in Theorem 2.2.4 (so that for every linear function $\gamma\colon L \to \mathbb{Z}$, $\gamma \in |\Sigma|$, zero is a regular value of $\varphi^\gamma = (\varphi_1|_{A_1^\gamma}, \dots, \varphi_k|_{A_k^\gamma})\colon T \to \mathbb{C}^k$) form a Zariski open subset of $\mathbb{C}^{A_1} \oplus \dots \oplus \mathbb{C}^{A_k}$.*

3) *Given $A_1, \dots, A_k$, suppose that Σ is a fan satisfying the assumptions of part 1), except possibly for completeness, and $V = \{\varphi_1 = \dots = \varphi_k = 0\}$ is a set satisfying the assumptions of part 2). Then the closure of V in the smooth toric variety X_Σ is a smooth subvariety transversal to all orbits in X_Σ.*

4) *Under the hypotheses of part 3), the intersection of the closure of V with the orbit $T_C = T/\langle t^l,\ l \in C\rangle$, $C \in \Sigma$, coincides with $\{\varphi^\gamma = 0\}/\langle t^l,\ l \in C\rangle$.*

Part 1) of this theorem is rather non-trivial but purely combinatorial (see [33]). The covering of X_Σ by affine charts reduces the other parts to the case of the affine toric variety $X_\Sigma = X_Q = \mathbb{C}^n$, where $Q \subset \mathbb{Q}^n$ is the positive octant and $0 \in A_i \subset Q$. In this case, the desired assertions follow from Sard's theorem for $\varphi = (\varphi_1, \dots, \varphi_k)\colon (\mathbb{C}\setminus\{0\})^n \to \mathbb{C}^k$. Indeed, this map extends to $\tilde{\varphi}\colon \mathbb{C}^n \to \mathbb{C}^k$, and the closure of $V \subset (\mathbb{C}\setminus\{0\})^n$ in $\mathbb{C}^n$ is given by $\tilde{\varphi} = 0$. Hence, by Sard's theorem, a generic point $c \in \mathbb{C}^k$ is a regular value of the restrictions of $\tilde{\varphi}$ to all coordinates planes. Parts 2)–4) of the theorem follow since

– the polynomial $\widetilde{\psi} := \tilde{\varphi} - c$ is also contained in $\mathbb{C}^A$;

– the equation $\widetilde{\psi} = 0$ determines a smooth subvariety, which intersects each coordinate plane in $\mathbb{C}^n$ transversally;

– the restriction of $\widetilde{\psi}$ to a coordinate plane is equal to $\widetilde{\psi}^\gamma$ for a covector γ in the corresponding coordinate plane of $\mathbb{Z}^n$.

Theorem 2.5.1 enables us to prove all the assertions about Newton polytopes in § 2.2 by means of toric geometry.

For an arbitrary algebraic set $V \subset (\mathbb{C}\setminus\{0\})^n$, a smooth toric compactification need not exist, but there is always a weaker version in the following sense.

Definition 2.5.2. 1) The truncation of a Laurent polynomial $f = \sum_{a\in A} c_a x^a$ in the direction of $l \in L^*$ is the sum of its leading monomials in the sense of the grading $l\colon L \to \mathbb{Z}$, that is, $f^l = \sum_{l(a)=l_0} c_a x^a$ for the largest l_0 such that this sum is non-zero.

2) The truncation of an ideal $I \subset \mathbb{C}[L]$ in the ring of Laurent polynomials is the ideal I^l generated by the truncations f^l, $f \in I$.

3) The truncation V^l of an algebraic variety $V \subset T$ in the direction of $l \in L^*$ is the zero variety of the ideal I^l, where I is the ideal of V.

Assertion 2.5.3. 1) *For every subvariety V of the complex torus T there is a complete non-singular fan Σ in the lattice L^* such that the closure of V in the toric variety X_Σ intersects each orbit T_C, $C \in \Sigma$, along a set of codimension* $\operatorname{codim} X_C + \operatorname{codim} V$.

2) *For every cone C of such a fan Σ, the truncations of V along all $l \in C$ coincide* (*we denote them by V_C*), *and therefore V_C is invariant under the action of all one-parameter subgroups t^l, $l \in C$.*

3) *The intersection of the closure of V with an orbit $T_C = T/\langle t^l,\ l \in C\rangle$ is equal to $V_C/\langle t^l,\ l \in C\rangle$.*

A proof will be given in § 5. Such a pair $(X_\Sigma, \overline{V})$ is called a tropical compactification of V. Following [62], we say that a smooth variety V is schön if it has a tropical compactification such that $\overline{V}$ is smooth. In particular, we have seen that all varieties given by sufficiently generic equations are schön.

3. Tropical geometry and A-discriminants

3.1. Tropical geometry. Tropical numbers are usually defined as elements of the set $\mathbb{R}$ endowed with $-\infty$ and the structure of a semiring, where the sum of a and b is $\max(a, b)$ and their product is $a + b$. However, we follow the ideas in [66] to give another definition of the sum.

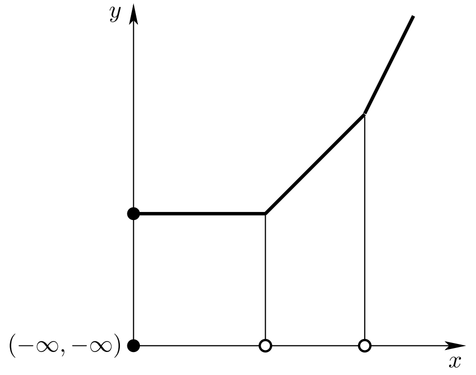

Figure 1. The graph of a tropical polynomial.

Let $\mathbb{T} = \mathbb{R} \sqcup \{-\infty\}$ be the tropical semifield with operations

$$a \underset{\mathbb{T}}{\cdot} b = a \underset{\mathbb{R}}{+} b \quad \text{and} \quad a \underset{\mathbb{T}}{+} b = \begin{cases} \max(a,b) & \text{if } a \neq b, \\ [-\infty, a] & \text{if } a = b. \end{cases}$$

This multivalued tropical sum is slightly more convenient than the usual one (see, for example, Definition 3.1.2 below). It is also more natural since the tropical sum and product are meant to axiomatize the behaviour of the degree of the sum and product of two polynomials, and we know that if $\deg f = \deg g$, then $\deg(f+g)$ can take any value in the interval $[-\infty, \deg f]$.

In what follows we always write $\mathfrak{1} \in \mathbb{T}$ and $\mathfrak{0} \in \mathbb{T}$ for $0 \in \mathbb{R}$ and $-\infty$, respectively. Moreover, since the sum is multivalued, '$= \mathfrak{0}$' in the tropical sense means '$\ni -\infty$'.

Remark 3.1.1. We denote tropical objects by Fraktur letters because we regard the tropical torus $\mathbb{T} \setminus \mathfrak{0}$ as the real part of the Lie algebra of the complex torus $\mathbb{C} \setminus \{0\}$.

Definition 3.1.2. A *tropical hypersurface* is a set of the form $S = \{x \mid f(x) = \mathfrak{0}\}$, where f is a tropical polynomial. In a sufficiently small neighbourhood of a generic point $x_0 \in S$, one can represent f in the form $g \cdot h^k$, where $g(x_0) \neq \mathfrak{0}$ and h is an irreducible polynomial. The greatest possible integer k with this property is called the *multiplicity* of S at x_0.

Moreover, in a small neighbourhood of a generic point $x_0 \in S$, the set $S \subset \mathbb{T}^n$ coincides with a hyperplane and the polynomial f is equal to its monomials $c_a x^a$ and $c_b x^b$ on the two sides of this hyperplane. Then $b - a \in \mathbb{Z}^n$ splits into a product $k \cdot v$, where $v \in \mathbb{Z}^n$ is a primitive vector and $k \in \mathbb{N}$ is called the multiplicity of S at x_0. (This coincides with the original definition since f is equal to $c_a x^a((c_b/c_a)x^v + 1)^k$ as a function in a neighbourhood of x_0.)

For example, Fig. 1 shows the graph of $f(x) = x^3 + x^2 + 6x + 8$ as well as the set $\{f = \mathfrak{0}\} = \{2, 3\}$ with multiplicity 1 and 2, respectively (empty circles).

Algebraic geometry over tropical numbers is useful because the answers to many enumerative problems appear to be the same over $\mathbb{C}$ and over $\mathbb{T}$. For example, we state a tropical analogue of the Kushnirenko–Bernstein theorem.

We say that tropical hypersurfaces $H_1, \dots, H_n$ in the tropical torus $\mathbb{R}^n = (\mathbb{T}\setminus \mathbb{o})^n$ intersect *transversally* if their intersection consists of finitely many points and, in a sufficiently small neighbourhood of every such point p, each hypersurface H_i looks like a hyperplane, that is, H_i is given (with multiplicity accounted for) by a tropical equation of the form $x^{a_{p,i}} + p^{a_{p,i}} = \mathbb{o}$ for some $a_{p,i} \in \mathbb{Z}^n$. Then the *intersection number* of $H_1, \dots, H_n$ at p is set to be $|\det(a_{p,1}, \dots, a_{p,n})|$.

Theorem 3.1.3 (tropical Kushnirenko–Bernstein theorem). *The total intersection number of transversal hypersurfaces $H_1, \dots, H_n$ is equal to the mixed volume of the Newton polytopes of their equations.*

Note that in the case when the equations of the hypersurfaces are of the form $x^{a_{p,i}} + p^{a_{p,i}} = \mathbb{o}$, this theorem coincides with the definition of the intersection number. (Hence the theorem could not hold for any other definition of the intersection number.)

Theorem 3.1.3 admits the following generalization to the case of non-transversal hypersurfaces. In this form, the theorem turns out to be an additivity property for mixed volumes and enables one to calculate the mixed volumes of complicated polytopes in terms of those of their simple parts.

Definition 3.1.4. The local Newton polytope of a tropical polynomial φ at a point x_0 is the minimally possible Newton polytope of a tropical polynomial which coincides with φ as a function in a neighbourhood of x_0.

We denote it by N_{φ,x_0}. More constructively, it is the convex hull of all a such that the value of the monomial $c_a x^a$ of the polynomial $\varphi(x) = \sum c_a x^a$ at the point $x = x_0$ is equal to the maximum value $\varphi(x_0)$.

Note that every polynomial φ has finitely many local Newton polytopes. They form a subdivision of its Newton polytope, which is called the *dual subdivision* of φ and is denoted by S_φ.

Theorem 3.1.5 (additivity of mixed volume under subdivision). *The sum over all $x \in \mathbb{T}^n$ of the mixed volumes of the local Newton polytopes $N_{\varphi_1,x}, \dots, N_{\varphi_n,x}$ is equal to the mixed volume of the Newton polytopes of the polynomials $\varphi_1, \dots, \varphi_n$.*

In particular, the mixed volumes of the local Newton polytopes vanish at all but finitely many points x due to dimensional considerations. In the case of transversal hypersurfaces, this theorem becomes the previous one. Hence it can naturally be restated in the following form.

Definition 3.1.6. The intersection number of arbitrary tropical hypersurfaces $\varphi_i = 0$ at a point x is the mixed volume of the local Newton polytopes $N_{\varphi_1,x}, \dots, N_{\varphi_n,x}$.

Note that when we perturb tropical hypersurfaces to make them transversal, their point of intersection x splits into several close points of transversal intersection x_i, and the intersection number at x (in the sense just defined) is, by Theorem 3.1.3, equal to the sum of the intersection numbers at x_i (in the sense defined before Theorem 3.1.3).

In this terminology, the last theorem takes the following form.

Theorem 3.1.7 (generalized tropical Kushnirenko–Bernstein theorem). *The total intersection number of tropical hypersurfaces $H_1, \dots, H_n$ (not necessarily transversal) is equal to the mixed volume of the Newton polytopes of their equations.*

In practice, it is convenient to construct the dual subdivision of the Newton polytope N of a polynomial $\varphi(x) = \sum_{a \in N \cap \mathbb{Z}^n} c_a x^a$ by using its *Legendre transform* $\tilde{\varphi}\colon N \to \mathbb{R}$. It is defined as the minimal upper-convex function whose value at each point $a \in N \cap \mathbb{Z}^n$ is not smaller than c_a.

Assertion 3.1.8. *The n-dimensional components of the dual subdivision of the Newton polytope N of a tropical polynomial φ are precisely the maximal domains of linearity of the Legendre transform $\tilde{\varphi}\colon N \to \mathbb{R}$.*

Example 3.1.9. The dual subdivision of the polynomial $5 + 10x + 11y + 6xy$ is the partition of its Newton polytope $[0,1]^2$ into two triangles by the diagonal from $(1,0)$ to $(0,1)$. The Legendre transform of this polynomial is a continuous function on $[0,1]^2$, which is linear on these triangles and takes the values 5, 10, 11, and 6 at their vertices.

The polynomial φ cannot be uniquely recovered from its Legendre transform $\tilde{\varphi}$ as a polynomial, but it can as a function $\varphi = \tilde{\tilde{\varphi}}$. In particular, $\tilde{\varphi}$ determines the hypersurface $\varphi = 0$ uniquely. For example, by the assertion above, its vertices are in a one-to-one correspondence with the domains of linearity of $\tilde{\varphi}$, and each vertex is equal to the differential of $\tilde{\varphi}$ on the corresponding domain of linearity.

Example 3.1.10. For all $c \leqslant 3$ the polynomial $2 + cx + 4x^2 \in \mathbb{T}[x]$ determines the same function $\mathbb{T} \to \mathbb{T}$ and has the same zero and the same Legendre transform. By contrast, for all $c \geqslant 3$ the Legendre transforms, the functions, and their zeros are pairwise different. However, the zeros can be uniquely recovered from the Legendre transforms in either case.

The relationship between the dual subdivision and the Legendre transform motivates the following definition.

Definition 3.1.11. A subdivision of a lattice polytope is said to be *coherent* (or convex or regular) if it is dual to some tropical polynomial or, equivalently, it is the set of domains of linearity of some upper-convex piecewise linear function.

Not all subdivisions are coherent.

3.2. The tropical correspondence theorem. The tropical Kushnirenko–Bernstein theorem is a simple instance of the following *tropical correspondence* principle, whose full generality is currently rather mysterious.

> The answers to many problems of computational algebraic geometry are the same over $\mathbb{C}$ and over $\mathbb{T}$.

This principle is useful because algebraic geometry over $\mathbb{T}$ is piecewise linear combinatorics. Therefore, a proof of the tropical correspondence principle for a given class of computational questions is by itself a combinatorial answer to these questions.

The first substantial class of questions for which this principle has been established is about computing the Gromov–Witten invariants of the projective plane.

Many proofs of this fact are known these days. The original analytic proof [47] used the language of amoebas, the proofs in [53] and [55] were oriented towards deformation theory, and those in [63] and [22] towards intersection theory. There are also many generalizations such as counting curves with more complicated singularities [54] which satisfy more general incidence conditions [5] or lie in spaces of higher dimension [55].

Definition 3.2.1. We write $N_{d,g}$ for the number of algebraic curves of degree d and genus g in $\mathbb{CP}^2$ passing through a fixed set of generic points $p_1, \dots, p_k$.

Clearly, this number can be finite and non-zero only if k is equal to the dimension $3d-1-g$ of the family of all plane curves of degree d and genus g (the so-called *Severi variety*). Therefore, we always assume that k is equal to $3d-1-g$.

A Zariski open subset of the Severi variety is formed by *nodal* curves, that is, curves with only the simplest singularities, namely transversal intersections (*nodes*). Hence the family of non-nodal curves of degree d and genus g has a strictly lower dimension and, therefore, all the curves in Definition 3.2.1 are nodal for $k = 3d - 1 - g$. Since the genus of a curve of degree d with n nodes is $(d-1)(d-2)/2 - n$, each curve in Definition 3.2.1 has $(d-1)(d-2)/2 - g$ nodes.

Definition 3.2.2. Let $\varphi \in \mathbb{T}[x, y]$ be a polynomial with Newton polytope N. The tropical curve $\varphi = \mathsf{o}$ is said to be nodal if the following conditions hold:

– a small neighbourhood of each of its non-smooth points x coincides with the union of three rays outgoing from x or two lines passing through x, that is, the dual subdivision of N consists of triangles and parallelograms;

– each of its rays is of multiplicity 1, that is, all the lattice points on the boundary of N are vertices of the dual subdivision.

The *number of nodes* of the curve $\varphi = \mathsf{o}$ is the sum

$$\begin{aligned}&(\text{the number of parallelograms in the dual subdivision})\\ &\quad + (\text{the number of lattice points in the Newton polytope } N\\ &\qquad \text{which are not vertices of the dual subdivision}).\end{aligned}$$

Correspondingly, the *genus* is the difference

$$\begin{aligned}&(\text{the number of vertices of the dual subdivision inside } N)\\ &\quad - (\text{the number of parallelograms in the dual subdivision}).\end{aligned}$$

The *multiplicity* of the curve $\varphi = \mathsf{o}$ is defined to be the product of lattice areas of the triangles in the dual subdivision of the Newton polytope of φ. The tropical Gromov–Witten invariant $N^{\mathbb{T}}_{d,g}$ is the total multiplicity of tropical curves of degree d and genus g passing through $3d-1-g$ fixed generic points.

Theorem 3.2.3 (Mikhalkin's theorem). *The equality $N_{d,g} = N^{\mathbb{T}}_{d,g}$ holds.*

Below we deduce the simplest case of this theorem (for curves with one node) from a description of the Newton polytope of the discriminant given by Gelfand, Kapranov, and Zelevinsky. Thus we begin by recalling this description and outlining a more elementary proof of the result than in the original book [21].

3.3. Support functions and secondary polytopes. Here we define a combinatorial object, the *secondary polytope*, which will later turn out to be the Newton polytope of the discriminant. We also define its mixed analogue, which will turn out to be the Newton polytope of the resultant.

The secondary polytope is most naturally defined in terms of its support function. We recall that the support function of a polytope N in a vector space V is the continuous piecewise linear function $N(\,\cdot\,)\colon V^* \to \mathbb{R}$ defined by $N(\gamma) = \max \gamma|_N$ for every linear function $\gamma\colon V \to \mathbb{R}$.

Remark 3.3.1. (i) A polytope is uniquely determined by its support function. More precisely, the vertices v_i of the polytope $N \subset V$ are in a one-to-one correspondence with the domains of linearity $L_i \subset V^*$ of the support function $N(\,\cdot\,)$, and $v_i \in V$ is equal to the linear function $N(\,\cdot\,)|_{L_i} \in V^{**} = V$.

(ii) These facts are particular cases of Assertion 3.1.8 and the subsequent discussion since the support function of a lattice polytope N can be written as a tropical polynomial $\varphi_N = \sum\limits_{a \in N \cap \mathbb{Z}^n} x^a$ with Newton polytope N.

In particular, the tropical hypersurface $\varphi_N = 0$ is important. It consists of the dual cones to the edges of N (see Definition 2.4.1), and the weight of each cone is equal to the lattice length of the corresponding edge.

Definition 3.3.2. The tropical hypersurface $\varphi_N = 0$ is called the dual tropical fan of the polytope N and is denoted by $[N]$. (This is not to be confused with the dual fan; see Definition 2.4.1.)

Definition 3.3.3. The secondary polytope of a set $A \subset \mathbb{Z}^n$ is the polytope $S_A \subset \mathbb{R}^A$ the value of whose support function on a covector $\gamma \in \mathbb{R}^A$ with coordinates γ_a, $a \in A$, is equal to the integral of the minimal upper convex function $\widetilde{\gamma}\colon \operatorname{conv} A \to \mathbb{R}$ such that $\widetilde{\gamma}(a) \geqslant \gamma_a$ for all $a \in A$.

Example 3.3.4. Consider the secondary polytope of the set $\{0,1,2\} \subset \mathbb{Z}^1$. The value of its support function on a covector $(\gamma_0, \gamma_1, \gamma_2) \in \mathbb{R}^{\{0,1,2\}}$ is $\gamma_0/2 + \gamma_1 + \gamma_2/2$ if $\gamma_1 \geqslant (\gamma_0 + \gamma_2)/2$ and $\gamma_0 + \gamma_2$ otherwise. Hence this secondary polytope is the closed interval with endpoints $(1/2, 1, 1/2)$ and $(1, 0, 1)$.

To describe the vertices of the secondary polytope using Remark 3.3.1, (i), we need to describe the domains of linearity of its support function. Note that under a variation $\gamma_a \rightsquigarrow \gamma_a + \varepsilon$ the integral of $\widetilde{\gamma}$ depends linearly on ε if all the domains of linearity S_i of $\widetilde{\gamma}$ adjacent to a are simplexes and, when $a \in S_i$, the equality $\widetilde{\gamma}(b) = \gamma_b$ holds only for the vertices $b \in S_i$. In this case the derivative of $\int_{\operatorname{conv} A} \widetilde{\gamma}(x)\,\mathrm{d}x$ with respect to ε is equal to $\dfrac{1}{n+1} \sum\limits_i \operatorname{Vol} S_i$. Otherwise the integral $\int_{\operatorname{conv} A} \widetilde{\gamma}(x)\,\mathrm{d}x$ as a function of ε exhibits a kink at $\varepsilon = 0$ and therefore has no derivative. We arrive at the following conclusion.

Assertion 3.3.5. 1) *The support function of the secondary polytope S_A is linear in a neighbourhood of $\gamma \in \mathbb{R}^A$ if and only if all domains of linearity of $\widetilde{\gamma}\colon \operatorname{conv} A \to \mathbb{R}$ are simplexes and the equality $\widetilde{\gamma}(a) = \gamma_a$ holds only when a is a vertex of one of them.*

2) *There is a one-to-one correspondence between the vertices of* S_A, *the domains of linearity of the support function* $S_A(\,\cdot\,)$, *and the coherent subdivisions of* $\operatorname{conv} A$ *into simplexes* (*that is*, *the triangulations*).

Under this correspondence, *every triangulation* $\{T_i\}$ *is associated with the vertex of* $\mathbb{R}^A$ *whose* a-*coordinate is equal to* $\dfrac{1}{n+1}\displaystyle\sum_{i\,|\,a\text{ is a vertex of }T_i}\operatorname{Vol}T_i$ *for every* $a\in A$. *The corresponding domain of linearity of* $S_A(\,\cdot\,)$ *consists of all* γ *such that* $\widetilde{\gamma}$ *is a linear function on each simplex* T_i.

In what follows we shall need a tropical interpretation of the support function $S_A(\,\cdot\,)$ and the function $\widetilde{\gamma}$ occurring in its definition. We regard the ambient space of the secondary polytope $\mathbb{T}^A\supset S_A$ as the space of tropical polynomials of the form $\gamma(x)=\displaystyle\sum_{a\in A}\gamma_a x^a$.

Remark 3.3.6. The function $\widetilde{\gamma}$ in Definition 3.3.3 is the Legendre transform of the tropical polynomial γ.

Note that the homothety $(n+1)!\,S_A$ is a lattice polytope. By Remark 3.3.1, (ii), the support function $(n+1)!\,S_A(\,\cdot\,)$ can be regarded as a tropical polynomial $\mathfrak{S}_A$ with Newton polytope $(n+1)!\,S_A$. The corresponding hypersurface $\mathfrak{S}_A=\mathfrak{o}$ is called the *secondary fan* of A. This set consists of all $\gamma\in\mathbb{T}^A$ that violate one of the conditions in Assertion 3.3.5, 1). When $n=2$, all such γ can be described explicitly.

Example 3.3.7. When $n=2$, the secondary fan $\mathfrak{S}_A=\mathfrak{o}$ is the union of the closures of the following cones C_T and $C_{T,a}$ of codimension 1:

1) T is an arbitrary coherent triangulation of the polytope $\operatorname{conv} A$ with vertices in A. The point $a\in A$ is not a vertex of T. The cone $C_{T,a}$ consists of all γ such that the elements of T are domains of linearity of $\widetilde{\gamma}$ and $\widetilde{\gamma}(a)=\gamma_a$.

2) T is an arbitrary coherent subdivision of the polytope $\operatorname{conv} A$ with vertices in A such that one element of T is a quadrangle and the other elements are triangles. The cone C_T consists of all γ such that the elements of T are domains of linearity of $\widetilde{\gamma}$.

3.4. Resultants and discriminants.

3.4.1. *Resultants.* Given a finite set $A\subset\mathbb{Z}^n$, we write $(\mathbb{C}\setminus\{0\})^A$ for the space of Laurent polynomials of the form $\displaystyle\sum_{a\in A}c_a x^a$, $c_a\neq 0$.

Definition 3.4.1. The A-*resultant*

$$R_A\subset\underbrace{(\mathbb{C}\setminus\{0\})^A\times\cdots\times(\mathbb{C}\setminus\{0\})^A}_{n+1}$$

is the closure of the set of all $(n+1)$-tuples of polynomials having a common root in $(\mathbb{C}\setminus\{0\})^n$.

Speaking of A-resultants, we always assume that A does not lie in any affine hyperplane. Then the resultant is an irreducible algebraic hypersurface. We denote its defining equation by R_A again.

Example 3.4.2. When $A = \{0, 1, \dots, d\} \subset \mathbb{Z}^1$, we obtain the classical resultant of two polynomials of equal degree in one variable.

We can make this definition more constructive by discarding the closure.

Assertion 3.4.3. *An $(n+1)$-tuple of polynomials $(f_0, \dots, f_n)$ is contained in R_A if and only if there is a face Γ of the convex hull of A such that the restrictions $f_0\big|_\Gamma, \dots, f_n\big|_\Gamma$ have a common root in $(\mathbb{C} \setminus \{0\})^n$.*

In fact, the polynomials $f_i = \sum c_{a,i} x^a$ have a common root if and only if the plane $\sum c_{a,i} y_a = 0$, $i = 0, \dots, n$, intersects the image of the embedding $j_A \colon (\mathbb{C} \setminus \{0\})^n \to \mathbb{CP}^A$. Hence the $(n+1)$-tuple of $f_i = \sum c_{a,i} x^a$ belongs to R_A if and only if the plane $\sum c_{a,i} y_a = 0$, $i = 0, \dots, n$, intersects the closure of the image, that is, the toric variety $X_A \subset \mathbb{CP}^A$. But X_A consists of orbits which are the images of $j_\Gamma \colon (\mathbb{C} \setminus \{0\})^n \to \mathbb{CP}^A$ over all faces $\Gamma \subset A$, and the intersection of this plane and orbit is non-empty if and only if the restrictions $f_0\big|_\Gamma, \dots, f_n\big|_\Gamma$ have a common root.

To describe the Newton polytope of the resultant, it will be convenient to use the following notation. Given any covector γ in the positive octant of the lattice $\mathbb{Z}^A$, we define a polytope $N_\gamma \subset \mathbb{R}^{n+1}$ as the ordinate set of the non-negative function $\widetilde{\gamma}$ in Definition 3.3.3.

Lemma 3.4.4. *The value of the support function of the Newton polytope of R_A on a covector $\gamma = (\gamma_0, \dots, \gamma_n)$ in the positive octant of the lattice $\mathbb{Z}^A \oplus \dots \oplus \mathbb{Z}^A$ is equal to the mixed volume $(n+1)!\, \mathrm{MV}(N_{\gamma_0}, \dots, N_{\gamma_n})$.*

To prove this, we consider the curve

$$F(t) = (F_0(t), \dots, F_n(t)) = \Big(\sum c_{a,0}(t) x^a, \dots, \sum c_{a,n}(t) x^a\Big)$$

in the space

$$\underbrace{(\mathbb{C} \setminus \{0\})^A \times \dots \times (\mathbb{C} \setminus \{0\})^A}_{n+1},$$

where $c_{a,i}(t) \in \mathbb{C}[t]$ is a generic polynomial of degree $\gamma_{i,a}$. On the one hand, the degree of the polynomial $R_A(F(t)) \in \mathbb{C}[t]$ is equal to the value of the support function of the Newton polytope of the resultant on γ. On the other hand, the roots of $R_A(F(t))$ are in a one-to-correspondence with the roots of the generic system of equations $F_0(t)(x) = \dots = F_n(t)(x) = 0$ in $n+1$ variables. Therefore, by the Kushnirenko–Bernstein formula (Theorem 2.2.1), the number of such roots is equal to the mixed volume of the Newton polytopes $F_i(t)(x)$, that is, $(n+1)!\, \mathrm{MV}(N_{\gamma_0}, \dots, N_{\gamma_n})$.

Remark 3.4.5. The lemma just proved determines the Newton polytope of the resultant R_A uniquely. Indeed, since the resultant is homogeneous, the value of the support function of its Newton polytope at a point γ is equal to its value at any point of the form $\gamma + (d, d, \dots, d)$, $d \in \mathbb{R}$, with the last quantity described by Lemma 3.4.4 for any sufficiently large d.

3.4.2. *Discriminants.*

Definition 3.4.6. The principal A-discriminant is the Laurent polynomial on $(\mathbb{C}\setminus\{0\})^A$ given by

$$E_A(f)=R_A\biggl(f,x_1\frac{\partial f}{\partial x_1},\dots,x_n\frac{\partial f}{\partial x_n}\biggr).$$

The A-discriminant D_A is the closure of the set of all polynomials $f\in(\mathbb{C}\setminus\{0\})^A$ with critical value zero. If D_A is an irreducible hypersurface, then we denote its irreducible equation also by D_A (otherwise we denote the unit polynomial by D_A).

Assertion 3.4.3 can be restated for the principal A-discriminant in the following way: $E_A(f)=0$ if and only if there is a face $\Gamma\subset A$ such that the restriction $f\big|_\Gamma\colon(\mathbb{C}\setminus\{0\})^n\to\mathbb{C}$ has critical value zero, that is, $D_A(f|_\Gamma)=0$. Therefore, up to a monomial factor, the polynomial $E_A(f)$ is equal to the product over all the faces $\Gamma\subset A$ of the polynomials $D_A\bigl(f\big|_\Gamma\bigr)$ raised to some powers $c_A^\Gamma\in\mathbb{N}$. These powers were explicitly calculated in [21]. But if the toric variety X_A is smooth at the points of its Γ-orbit, then it readily follows by an argument similar to the proof of Assertion 3.4.3 that $c_A^\Gamma=1$. In particular, in low dimensions we obtain the following results.

Assertion 3.4.7. 1) *For $n=1$ we have $E_A=D_A$ up to a monomial factor.*

2) *Let $A\subset\mathbb{Z}^2$ be the set of lattice points of an integral polygon N. Then, up to a monomial factor, $E_A(f)=\prod\limits_\Gamma D_{A\cap\Gamma}\bigl(f\big|_\Gamma\bigr)$, where Γ ranges over N and all its sides.*

Theorem 3.4.8. 1) *The Newton polytope of E_A is equal to S_A. In particular, the degree of E_A is equal to $(n+1)!\operatorname{Vol}(\operatorname{conv}A)$.*

2) *Let $A\subset\mathbb{Z}^2$ be the set of lattice points of an integral polygon N. Then the Newton polytope N_{D_A} of the discriminant D_A is uniquely determined* (*up to a shift*) *by the equation $S_A=N_{D_A}+\sum\limits_\Gamma S_{A\cap\Gamma}$, where Γ ranges over all sides of N.*

To prove part 1) note that the support function of the Newton polytope of E_A at γ is $\displaystyle\int\widetilde{\gamma}(x)\,\mathrm{d}x$ by Lemma 3.4.4 and Remark 3.4.5. In particular, the degree of E_A is equal to the value of the support function at $(1,\dots,1)$.

Part 2) follows from part 1) and Assertion 3.4.7 since the Newton polytope of a product is equal to the sum of the Newton polytopes of the factors.

By Remark 3.3.1, (ii), the support function of the Newton polytope N_{D_A} can be regarded as a tropical polynomial with Newton polytope N_{D_A}. We call this polynomial the *tropical discriminant* and denote it by $\mathfrak{D}_A$. Then part 2) of the theorem can be restated as an equality for tropical polynomials:

$$\mathfrak{S}_A=\mathfrak{D}_A\cdot\prod_\Gamma\mathfrak{S}_{A\cap\Gamma}. \tag{3.1}$$

This equality implies that the tropical hypersurface $\mathfrak{D}_A=\mathfrak{o}$ is contained in $\mathfrak{S}_A=\mathfrak{o}$, that is, in the union of cones in Example 3.3.7. However, there is an important class of cones in Example 3.3.7 not contained in $\mathfrak{D}_A = \mathfrak{o}$. Namely, the following lemma holds.

Lemma 3.4.9. *Let $a \in N$ be the unique lattice point which is not a vertex of the triangulation T of the polygon N. Suppose that a lies inside a side of the Newton polygon $\Gamma_0 \subset N$. Then the interior of $C_{T,a}$ is disjoint from $\mathfrak{D}_A = \mathfrak{o}$.*

Indeed, a is the unique interior lattice point (hence the midpoint) of a side $I \subset \Gamma_0$ of a triangle in the triangulation. Hence the polynomials $\mathfrak{D}_A$ and $\mathfrak{S}_{\mathbb{Z}^2 \cap \Gamma_0}$ coincide in a neighbourhood of an interior point $\gamma \in C_{T,a}$ with the polynomial $\mathfrak{S}_{\mathbb{Z}^2 \cap I}$ (which was described in Example 3.3.4). Correspondingly, (3.1) takes the following form in a neighbourhood of γ:

$$\mathfrak{S}_{\mathbb{Z}^2 \cap I} = \mathfrak{D}_A \cdot \mathfrak{S}_{\mathbb{Z}^2 \cap I} \cdot (\text{a tropical monomial}).$$

Hence in a neighbourhood of γ the polynomial $\mathfrak{D}_A$ coincides with a tropical monomial (in customary terminology, a linear function), and therefore it has no tropical roots near γ.

3.5. Proof of the correspondence theorem in the simplest case. Let A be the set of lattice points in an integral polygon $N \subset \mathbb{R}^2$, $0 \in N$. We want to calculate the number of polynomials $f \in \mathbb{C}^A$ such that the curve $f = 0$ has one singularity and passes through a fixed q-tuple of generic points $p_1, \dots, p_q \in (\mathbb{C} \setminus \{0\})^2$, where $q = |A| - 2$. In other words, we want to calculate the intersection number I of the following hypersurfaces in $\mathbb{C}^A$:

– the incidence conditions $H_1, \dots, H_q$, where $H_i = \{f \mid f(p_i) = 0\}$;
– the normalization $H_0 = \{f \mid f(0) = 1\}$;
– the A-discriminant $D_A = \{f \mid f = 0 \text{ is not regular}\}$.

We tropicalize these objects, choose points $\mathfrak{p}_1, \dots, \mathfrak{p}_q \in (\mathbb{T} \setminus \{0\})^2$, and define the tropical hypersurfaces

$$\mathfrak{H}_i = \{f \mid f(\mathfrak{p}_i) = \mathfrak{o}\} \quad \text{in } \mathbb{T}^A \quad \text{and} \quad \mathfrak{H}_0 = \{f \mid f(\mathfrak{o}) = \mathfrak{1}\}.$$

If the points $\mathfrak{p}_1, \dots, \mathfrak{p}_q$ are generic, then the tropical hypersurfaces $\mathfrak{H}_0, \dots, \mathfrak{H}_q$, and $\mathfrak{D} = \mathfrak{o}$ intersect transversally. We denote their intersection number by $\mathfrak{I}$. Then

$$I = \mathfrak{I} \tag{3.2}$$

because both sides are equal to the mixed volume of the Newton polytopes of the hypersurfaces $H_0, \dots, H_q$ and D_A by the Kushnirenko–Bernstein formula over $\mathbb{C}$ and $\mathbb{T}$, respectively (Theorems 2.2.1 and 3.1.7).

We now identify those cones of the tropical hypersurface $\mathfrak{D} = \mathfrak{o}$ that can intersect generic incidence hypersurfaces $\mathfrak{H}_i$.

(a) If the cone C_T in Example 3.3.7 corresponds to a subdivision T of a polygon N all of whose lattice points are vertices, then all triangles in this subdivision have area 1/2 and the only quadrangle is a parallelogram with area 1. Hence, for every $\mathfrak{f} \in C_T$, the curve $\mathfrak{f} = \mathfrak{o}$ is a nodal curve with one node.

(b) If the cone C_T in Example 3.3.7 corresponds to a subdivision T of a polygon N such that not all of its lattice points are vertices (for instance, there is a non-vertex lattice point b), then, even though the dimension of the cone is equal to $|A| - 2$, the dimension of the corresponding family of tropical curves $\mathfrak{f} = \mathfrak{o}$, $\mathfrak{f} \in C_T$, is strictly lower since they are independent of the coefficient c_b of the polynomial $\mathfrak{f} = \sum_a c_a x^a$. Since a tropical curve in a family with less than $|A| - 2$ parameters cannot pass through $|A| - 2$ generic points, such a cone C_T is disjoint from the generic incidence hypersurfaces $\mathfrak{H}_i$.

(c) If the cone $C_{T,a}$ in Example 3.3.7 corresponds to a triangulation T of a polygon N such that not all of its lattice points other than a are vertices, then this cone is disjoint from the incidence hypersurfaces $\mathfrak{H}_i$ like in the previous case.

(d) If the cone $C_{T,a}$ in Example 3.3.7 corresponds to a triangulation T of a polygon N all of whose lattice points other than a are vertices, then the following two cases may occur in accordance with Lemma 3.4.9:

– the point a lies inside a triangle of this triangulation, and this triangle has area $3/2$ while the other triangles have area $1/2$;

– the point a lies inside an interior edge of this triangulation, and the two triangles adjoining this edge have area 1 while the others have area $1/2$.

In both cases, for every $\mathfrak{f} \in C_{T,a}$ the curve $\mathfrak{f} = \mathfrak{o}$ is nodal with one node.

Thus we have found that the intersection of the tropical hypersurfaces $\mathfrak{H}_0, \dots, \mathfrak{H}_q$, $\mathfrak{D} = \mathfrak{o}$ consists of exactly those polynomials $\mathfrak{f}$ for which the curve $\mathfrak{f} = \mathfrak{o}$ is nodal with one node and passes through the incidence points $\mathfrak{p}_1, \dots, \mathfrak{p}_q$. It also follows from the definitions that the multiplicity of intersection of these hypersurfaces at every such point $\mathfrak{f}$ is equal to the multiplicity of the nodal curve $\mathfrak{f} = \mathfrak{o}$ in the sense of the correspondence theorem. (Namely, it is 3 if $\mathfrak{f} \in C_{T,a}$ and a lies inside a triangle of the triangulation T, it is 4 if a lies on an interior edge of T, and otherwise it is 1.)

Therefore, the intersection number of the tropical hypersurfaces $\mathfrak{H}_0, \dots, \mathfrak{H}_q$, $\mathfrak{D} = \mathfrak{o}$ (as we recall, it coincides with the quantity to be found; see (3.2)), is equal to the number of nodal tropical curves with one node which pass through the fixed generic points, with multiplicity accounted for.

4. The ring of conditions of the complex torus and tropical fans

The ring of conditions (see [11]) is an intersection theory for algebraic cycles in spherical homogeneous spaces with coefficients in a commutative ring Λ. In this section we consider the ring of conditions $\mathscr{R}_n(\Lambda)$ of the group $(\mathbb{C}^*)^n$ (which is a spherical homogeneous space with respect to the action of this group on itself) with coefficients in $\Lambda = \mathbb{Z}, \mathbb{R}, \mathbb{C}$. Many aspects of the ring $\mathscr{R}_n(\Lambda)$ can be described in terms of the cohomology rings of smooth projective toric varieties [11]. Historically, the first model of this ring was the so-called polytope algebra [46], [8]. The ring can also be described using tropical geometry. Its model is the ring of tropical fans (see [19], [29], [48], [20], [25], [43]). We recall these known results.

4.1. Ring of conditions.

4.1.1. *The ring of conditions $\mathscr{R}_n(\Lambda)$ of $(\mathbb{C}^*)^n$.* We recall the definition of the ring of conditions (see § 1.4). Cycles $X_1, X_2 \subset (\mathbb{C}^*)^n$ of dimension k are *equivalent* ($X_1 \sim X_2$) if, for every cycle $Y \subset (\mathbb{C}^*)^n$ of dimension $n-k$ and for almost all $g \in (\mathbb{C}^*)^n$, we have $\langle X_1, gY\rangle = \langle X_2, gY\rangle$, where $\langle A, B\rangle$ is the intersection number of A and B. If $X_1 \sim X_2$ and $Y_1 \sim Y_2$, then $X_1 \cap g_1Y_1 \sim X_2 \cap g_2Y_2$ for almost all

$g_1, g_2 \in (\mathbb{C}^*)^n$. The *product* $X * Y$ of equivalence classes X and Y is the equivalence class of the intersection $X_1 \cap g_1 Y_1$, where X_1 and Y_1 are representatives of X and Y, and g_1 is a generic element of $(\mathbb{C}^*)^n$. The *ring of conditions* $\mathscr{R}_n(\Lambda)$ is the ring of equivalence classes with product $*$ and tautological sum.

4.1.2. *The Bergman cone.* A vector $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ is said to be *essential* for a variety $X \subset (\mathbb{C}^*)^n$ if there exists a meromorphic function germ $f\colon (\mathbb{C}, 0) \to X \subset (\mathbb{C}^*)^n$ such that $f(t) = at^k + \cdots$, $a = (a_1, \dots, a_n) \in (\mathbb{C}^*)^n$, and the dots stand for higher-order terms $a_m t^m$ (with $m = (m_1, \dots, m_n)$, where $m_i \geqslant k_i$ for $1 \leqslant i \leqslant n$ and $m \neq k$).

The *Bergman cone* $B(X) \subset \mathbb{R}^n$ in X is the closure of the set of vectors $\lambda k \in \mathbb{R}^n$, where k is an essential vector for X and $\lambda \geqslant 0$.

Theorem 4.1.1. *If every irreducible component of X has complex dimension l, then $B(X)$ is a finite union of convex rational cones $|\sigma_i| \subset \mathbb{R}^n$, where $\dim_{\mathbb{R}} |\sigma_i| = l$. Moreover, $B(X)$ can be subdivided into the fan of a toric variety.*

The first version of this theorem appeared in [3]. The development of this construction in the context of tropical algebraic geometry began, in particular, with [60] and [13].

4.1.3. *Good compactification.* A toric variety $M \supset (\mathbb{C}^*)^n$ is called a *good compactification* for a subvariety $X \subset (\mathbb{C}^*)^n$ with $\dim X = k$ if the closure $\overline{X}$ in M is complete and disjoint from the orbits in M of codimension greater than k.

Theorem 4.1.2. 1) *For every finite set $\mathscr{S}$ of algebraic subvarieties in $(\mathbb{C}^*)^n$ there exists a toric variety $M \supset (\mathbb{C}^*)^n$ giving rise to a good compactification for every subvariety in $\mathscr{S}$.*

2) *A toric variety M is a good compactification of a k-dimensional subvariety $X \subset (\mathbb{C}^*)^n$ if and only if the support of the k-dimensional skeleton of its fan contains the Bergman cone $B(X)$.*

The first part of this theorem was established in [11]. It is very important for the theory of rings of conditions and tropical geometry. A transparent geometric proof of this result can be found in [39]. There are a number of refinements of this theorem (for example, see [62]). The strongest of them has been announced in [39].

Let $\mathscr{S}_r$ be the subset in the set of all subvarieties of $(\mathbb{C}^*)^n$ such that every X in $\mathscr{S}_r$ can be given by a system of Laurent polynomials whose Newton polytopes lie in a ball of radius r. The following more precise version of part 1) of Theorem 4.1.2 follows easily from [32].

Theorem 4.1.3. *There is a Newton polytope Δ_r such that the projective toric variety M_{Δ_r} corresponding to Δ_r is smooth and gives rise to a good compactification for every $X \in \mathscr{S}_r$. For every $X \in \mathscr{S}_r$ its Bergman cone $B(X)$ is the support of a subfan in the dual fan of Δ_r.*

4.1.4. *The ring $\mathscr{R}_n(\Lambda)$ and the cohomology ring of a toric variety.* Given a complete toric variety $M \supset (\mathbb{C}^*)^n$ and a cycle $X = \sum k_i X_i$ of dimension k, one can define a cycle $\overline{X}$ in M as $\sum k_i \overline{X}_i$, where $\overline{X}_i$ is the closure of $X_i \subset (\mathbb{C}^*)^n$ in M. The cycle $\overline{X}$ determines an element $\rho(\overline{X})$ of $H^{2(n-k)}(M^n, \Lambda)$ whose value at the closure $\overline{O}_i$ of any orbit O_i of dimension $n-k$ in M is equal to the intersection number $\langle \overline{X}, \overline{O}_i \rangle$. A compactification $M \supset (\mathbb{C}^*)^n$ is said to be *good* for $X = \sum k_i X^i$ in $(\mathbb{C}^*)^n$ if it is good for every X_i.

Theorem 4.1.4 (see [11]). *If a smooth toric compactification M is good for cycles X, Y, and Z, where $Z = X * Y$ (the product in the ring of conditions), then the product $\rho(X)\rho(Y)$ of $\rho(X)$ and $\rho(Y)$ in the cohomology ring $H^*(M, \Lambda)$ is equal to $\rho(Z)$.*

In particular, this theorem shows that the embedding

$$H^\bullet(M, \Lambda) \hookrightarrow \mathscr{R}_n(\Lambda) \tag{4.1}$$

sending each cohomology class of a smooth toric variety to a representing cycle which intersects all the orbits properly is a ring homomorphism.

4.1.5. *On Chow rings of algebraic varieties.* The Chow ring of an algebraic variety is an algebraic analogue of the ring of intersections of a compact orientable manifold. Every algebraic subvariety $N \subset M$ is called a *basic cycle*. The degree of a basic cycle N is denoted by $\dim N$. The cycles in the Chow group of M are integer linear combinations of basic cycles modulo so-called rational equivalence. The product of basic cycles in the Chow group is defined to be the intersection of these cycles after they are made generic. This product extends to all cycles in the Chow group by linearity and endows the graded Chow group with the structure of a commutative ring.

The Chow group. To define the graded Chow group and the Chow ring more precisely, we begin with a definition of trivial basic k-cycles in M. Let W be a $(k+1)$-dimensional algebraic subvariety of M, let $\pi\colon \widetilde{W} \to W$ be the natural projection onto W of its normalization $\widetilde{W}$, let f be a rational function on $\widetilde{W}$, and let (f) be the principal divisor of this function on $\widetilde{W}$. The image of (f) under π is called a trivial basic k-cycle in M. A linear combination of trivial basic k-cycles is called a k-cycle rationally equivalent to zero.

The graded Chow group $A_*(M)$ of an n-dimensional algebraic variety M is the direct sum $A(M) = A_0(M) + A_1(M) + \cdots + A_n(M)$ of its Chow groups $A_k(M)$ of dimension $k = 0, 1, \ldots, n$, where $A_k(M)$ is the quotient group of all k-cycles modulo k-cycles rationally equivalent to zero. The group $A_*(M)$ is analogous to the homology group of M. One can also define a graded group $A^*(M) = A^0(M) + A^1(M) + \cdots + A^n(M)$, where $A^k(M) = A_{n-k}(M)$. It is analogous to the cohomology group of M.

For a compact M, there is a map $\rho\colon A_k(M) \to H_{2k}(M, Z)$ sending every linear combination of k-dimensional algebraic varieties to the corresponding linear combination of their fundamental cycles. This map is well defined since cycles rationally equivalent to zero correspond to cycles homologous to zero in M. The Chow group $A_k(M)$ can be considerably larger than $H_{2k}(M, \mathbb{Z})$.

Example 4.1.5. Let M be a smooth connected algebraic curve of genus r. Then the homomorphism $\rho\colon A_0 \to H_0(M, \mathbb{Z}) = \mathbb{Z}$ is surjective and its kernel is isomorphic to the Jacobian of M (which is a compact complex torus of complex dimension $2r$).

On the other hand, the Chow group of a variety with complicated topology may be small.

Example 4.1.6. Let X be a closed algebraic subvariety of $\mathbb{C}^n$. Then the Chow group $A_n(M)$ of the variety $M = \mathbb{C}^n \setminus X$ is isomorphic to $\mathbb{Z}$. It is generated by the fundamental cycle of this variety. The other groups $A_k(M)$ are equal to zero. Note that the topology of M depends on that of X and can be rather complicated.

The graded Chow group enjoys a particularly close connection with the topology in the class of algebraic CW-complexes (see Definition 6.2.5), which contains all smooth projective toric varieties.

Example 4.1.7. Let M be a smooth projective algebraic variety endowed with the structure of an algebraic CW-complex. Then its graded Chow group is isomorphic (up to grading) to its graded homology group with integer coefficients.

The Chow ring. Chow proved that the graded group $A^*(M)$ of a smooth quasiprojective variety M can be endowed with the structure of a commutative ring.

Definition 4.1.8. Cycles $X \in A^k(M)$ and $Y \in A^m(M)$ are said to be algebraically transversal if their intersection $X \cap Y$ is either empty or has codimension $k + m$ in M.

When X and Y are algebraically transversal, then each component of $X \cap Y$ has well-defined multiplicity. The product $X \cdot Y$ of algebraically transversal cycles X, Y is the formal sum of the components of $X \cap Y$ with multiplicity accounted for.

Theorem 4.1.9 (Chow's theorem). 1) *For any cycles $X \in A^k(M)$ and $Y \in A^m(M)$ there are algebraically transversal cycles X' and Y' which are rationally equivalent to X and Y, respectively.*

2) *If X', Y' and X'', Y'' are algebraically transversal pairs such that X', X'' and Y', Y'' are rationally equivalent, then $X' \cdot Y'$ and $X'' \cdot Y''$ are also rationally equivalent.*

Chow's theorem enables us to endow the graded Chow group $A^*(M)$ of a smooth quasiprojective variety M with the structure of a ring with multiplication $A^k(M) \times A^m(M) \to A^{k+m}(M)$ such that the product of algebraically transversal cycles X, Y is equal to $X \cdot Y$. Multiplication of Chow cycles is compatible with the operation of intersection of the corresponding homology classes.

Theorem 4.1.10. *Let M be a smooth projective algebraic variety with the structure of an algebraic CW-complex. Then the Chow ring of M, the intersection ring of integer homology classes, and the integer cohomology ring of M are isomorphic up to a change of grading.*

Thus, the Chow ring provides a purely algebraic version of the ring of intersections (or the cohomology ring) of a smooth toric variety. This version is defined not only for complex toric varieties but also for toric varieties over any algebraically closed field. On the other hand, for a complex toric variety, the intersection of any cycles (including real ones) is controlled by its ring of intersections.

Chow ring and the ring of conditions of the group $(\mathbb{C}\setminus\{0\})^n$. Linear combinations of algebraic subvarieties of $(\mathbb{C}\setminus\{0\})^n$ occur in the definition of the Chow ring as well as in the definition of the ring of conditions of this group. But these rings are very different. The Chow ring is isomorphic to the ring of integers, with the fundamental class of the group as its generator. By contrast, the ring of conditions of this group is very rich. In particular, it admits a version of Theorem 2.2.1, which makes it possible to calculate the number of points of intersection of n hypersurfaces determined by sufficiently generic equations with fixed Newton polytopes. It follows from the general results due to De Concini and Procesi that the ring of conditions is the projective limit of the Chow rings (or integer cohomology rings) of all possible smooth projective toric compactifications of $(\mathbb{C}\setminus\{0\})^n$. This result is based on the theorem of existence of a good compactification for every algebraic subvariety of $(\mathbb{C}\setminus\{0\})^n$.

De Concini and Procesi [11] defined the ring of conditions for any homogeneous *spherical* space. The rings of conditions can be regarded as a generalization of the classical Schubert calculus. The rings of conditions and Schubert calculus coincide for the flag varieties of reductive groups, which are compact homogeneous spaces. Unfortunately, the rings of conditions of non-compact spherical spaces are difficult to describe. Descriptions of such rings are currently known only for some spaces. They were found using equivariant cohomology. For the class of horispherical homogeneous spaces, which contains $(\mathbb{C}\setminus\{0\})^n$ and all flag varieties, there is a description of the ring of conditions which is close to the description of this ring for $(\mathbb{C}\setminus\{0\})^n$ presented in our survey; see [24].

4.2. The ring of tropical fans.

4.2.1. *The ring of balanced* Λ*-weighted fans.* In this subsection we construct an important combinatorial model of the ring of conditions of a complex torus and the cohomology rings of smooth toric varieties.

4.2.1.1. A Λ*-weighted* k*-fan* is the fan $\mathscr{F}\subset\mathbb{R}^n$ of a toric variety of dimension n endowed with a *weight function* $c\colon\mathscr{F}_k\to\Lambda$ on the set $\mathscr{F}_k$ of all cones of dimension k in $\mathscr{F}$. The *support* $|\mathscr{F}|$ of a fan $\mathscr{F}$ is the union of all cones $|\sigma_i|\subset\mathbb{R}^n$ such that $\sigma_i\in\mathscr{F}_k$ and $c(\sigma_i)\neq 0$. Weighted k-fans $\mathscr{F}_1$ and $\mathscr{F}_2$ are *equivalent* if 1) $|\mathscr{F}_1|=|\mathscr{F}_2|$ and 2) the weight functions c_1 and c_2 induce the same weight function on all the cones of a common subdivision of $\mathscr{F}_1$ and $\mathscr{F}_2$.

4.2.1.2. Let $\mathscr{F}$ be a weighted k-fan. Given any cone $\sigma_i\in\mathscr{F}_k$, we write $L_i^\perp\subset(\mathbb{R}^n)^*$ for the dual space to the ambient space L_i of the support $|\sigma_i|\subset\mathbb{R}^n$. Let O be an orientation of $|\sigma_i|$. We write $e_i^\perp(O)\in\Lambda^{n-k}L_i^\perp$ for a vector of dimension $n-k$ such that 1) the lattice volume $|e_i^\perp(O)|$ in $L_i^\perp$ is equal to 1, and 2) the orientation $e_i^\perp(O)$ is induced by the orientation O of $|\sigma_i|$ and the standard orientation of $\mathbb{R}^n$. A weighted cone of a k-fan $\mathscr{F}$ satisfies the *balancing condition* if the following relation holds for any orientation of the cone $|\rho|$ of dimension $k-1$, where $\rho\in\mathscr{F}_{k-1}$:

$$\sum e_i^\perp(O(\rho))c(\sigma_i)=0, \tag{4.2}$$

where c is the weight function, the sum is taken over all $\sigma_i\in\mathscr{F}_k$ with $|\rho|\subset\partial|\sigma_i|$, and $O(\rho)$ is an orientation of $|\sigma_i|$ such that the orientation of $\partial|\sigma_i|$ is compatible with that of $|\rho|$.

Remark 4.2.1. It is easily verifiable that a weighted cone of codimension 1 is balanced if and only if it is a tropical hypersurface in the sense of § 3.1. This explains why balanced weighted fans are also said to be tropical.

4.2.1.3. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be balanced k- and $(n-k)$-fans. Cones $\sigma_i^1 \in \mathscr{F}_1$ and $\sigma_j^2 \in \mathscr{F}_2$ with $\dim \sigma_i^1 = k$ and $\dim \sigma_j^2 = n-k$ are said to be *a-admissible* for a given vector $a \in \mathbb{R}^n$ if $|\sigma_i^1| \cap (|\sigma_j^2| + a) \neq \varnothing$. Let $C_{i,j}$ be the index of $\Lambda_i \oplus \Lambda_j$ in $\mathbb{Z}^n$, where $\Lambda_i = L_i^1 \cap \mathbb{Z}^n$, $\Lambda_j = L_j^2 \cap \mathbb{Z}^n$, and L_i^1, L_j^2 are the linear spaces spanned by $|\sigma_i^1|$, $|\sigma_j^2|$, respectively. The *intersection number* $c(0)$ of the fans $\mathscr{F}_1$ and $\mathscr{F}_2$ is defined as

$$\sum C_{i,j} c_1(\sigma_i^1) c_2(\sigma_j^2), \tag{4.3}$$

where the sum is taken over all a-admissible pairs σ_i^1, σ_j^2 for a generic vector $a \in \mathbb{R}^n$. (One can show that if $\mathscr{F}_1$ and $\mathscr{F}_2$ satisfy the balancing condition (4.2), then the sum (4.3) is independent of the choice of a.)

Remark 4.2.2. (i) This definition of the intersection number can be understood informally as follows. For any generic vector a, the shift of the second fan by a intersects the first fan transversally at a finite number of points. This enables us to endow every such transversal intersection point with a weight as in (4.3) and define the intersection number as the weighted number of intersection points.

(ii) This point of view makes clear an analogy with the definition of the ring of conditions, where we also shift one of the intersecting subvarieties by a generic element in order to define the intersection number. (However, we do not formalize this analogy.) In this sense, the ring structure on the space of tropical fans (to be introduced in the next subsection) can be regarded as an analogue of the ring of conditions in tropical algebraic geometry.

4.2.1.4. Consider a k-fan $\mathscr{F}_1$ and an m-fan $\mathscr{F}_2$ in the set $T\mathscr{R}_n(\Lambda)$ of all balanced Λ-weighted fans. Denote $n-(k+m)$ by d. If $d<0$, then $\mathscr{F}_1 \times \mathscr{F}_2 = 0$. If $d=0$, then the *product* $\mathscr{F} = \mathscr{F}_1 \times \mathscr{F}_2$ is the 0-fan $\mathscr{F} = \{0\}$ with weight $c(0)$ equal to the intersection number (defined above) of $\mathscr{F}_1$ and $\mathscr{F}_2$.

We now define the d-fan $\mathscr{F} = \mathscr{F}_1 \times \mathscr{F}_2$ for $d>0$. Suppose that $\mathscr{F}_1$ and $\mathscr{F}_2$ are subfans of a complete fan $\mathscr{G}$. Then $\mathscr{F} = \mathscr{F}_1 \times \mathscr{F}_2$ is also a subfan of $\mathscr{G}$. The weight $c(\delta)$ of a cone δ in $\mathscr{G}$ with $\dim \delta = d$ is defined as follows. Let L be the space spanned by $|\delta|$, and let $(\mathscr{F}_1)_\delta$ and $(\mathscr{F}_2)_\delta$ be weighted subfans of $\mathscr{F}_1$ and $\mathscr{F}_2$ containing all the cones of $\mathscr{F}_1$ and $\mathscr{F}_2$ which contain δ. The weight $c(\delta)$ of a cone δ in $\mathscr{F} = \mathscr{F}_1 \times \mathscr{F}_2$ is equal to the intersection number of the quotient images $(\mathscr{F}_1)_\delta$ and $(\mathscr{F}_2)_\delta$ in the quotient space $\mathbb{R}^n/L$ endowed with the quotient lattice $\mathbb{Z}^n/(L \cap \mathbb{Z}^n)$. (Note that these images are of complementary dimensions in the quotient space, so that their intersection number is well defined.)

Remark 4.2.3. In particular, this definition enables us to define the intersection number of several fans $\mathscr{F}_i$ of total codimension n as the multiplicity of a zero-dimensional fan, namely their product. As can easily be seen from the definitions, when the codimensions of all the fans are equal to 1 (so that the fans are tropical hypersurfaces), this intersection number coincides with that introduced in § 3.1.

4.2.2. *Tropicalization of the ring $\mathscr{R}_n(\Lambda)$.* Let M_Δ be the smooth complete toric variety constructed from an n-dimensional polytope Δ, and let $\Delta^\perp$ be its dual fan. We consider the ring $T\mathscr{R}_n(\Lambda,\Delta) \subset T\mathscr{R}_n(\Lambda)$ of balanced Λ-weighted fans which are Λ-linear combinations of cones in $\Delta^\perp$. The following theorems were proved in [19].

Theorem 4.2.4 [19]. *The ring $T\mathscr{R}_n(\Lambda,\Delta)$ is isomorphic to the cohomology ring $H^\bullet(M_\Delta,\Lambda)$. The component of $T\mathscr{R}_n(\Lambda,\Delta)$ containing k-fans corresponds to $H^{2n-2k}(M_\Delta,\Lambda)$ under this isomorphism.*

Theorem 4.2.5 [19]. *The ring of conditions $\mathscr{R}_n(\Lambda)$ is isomorphic to the tropical ring $T\mathscr{R}_n(\Lambda)$ of all Λ-weighted fans. (The rings $\mathscr{R}_n(\mathbb{Z})$, $\mathscr{R}_n(\mathbb{C})$ have similar descriptions.)*

In particular, since the embeddings $T\mathscr{R}_n(\Lambda,\Delta) \subset T\mathscr{R}_n(\Lambda)$ over all Δ exhaust the ring $T\mathscr{R}_n(\Lambda)$, the embeddings $H^\bullet(M,\Lambda) \hookrightarrow \mathscr{R}_n(\Lambda)$ (mentioned above; see (4.1)) over all smooth complete toric varieties M exhaust the ring of conditions.

Our next aim is to describe the isomorphisms in these two theorems explicitly. The isomorphism involving the cohomology ring is obvious: a class $\alpha \in H^{2n-2k}(M,\Lambda)$ is mapped to the tropical fan consisting of the k-dimensional cones C in $\Delta^\perp$ with weights $\alpha \cdot T_C$. (We recall that T_C is the orbit corresponding to C in the toric variety.)

The isomorphism between the ring of conditions and the ring of tropical fans requires a longer discussion, which occupies the next two subsections.

4.2.3. *The Kushnirenko–Bernstein theorem and the ring $\mathscr{R}_n(\Lambda)$.* Let $\{\Gamma_i\}$ be a family of n hypersurfaces in $(\mathbb{C}^*)^n$ given by the equations $P_i = 0$, where P_i are Laurent polynomials with Newton polytopes Δ_i. The Kushnirenko–Bernstein theorem (Theorem 2.2.1) can be stated as follows.

Theorem 4.2.6. *The intersection number of the hypersurfaces Γ_i in the ring of conditions is equal to the lattice mixed volume of the polytopes $\Delta_1,\dots,\Delta_n$.*

Let $\mathscr{F}_i$ be the tropical $(n-1)$-fan dual to Δ_i (see Definition 3.3.2). By Remark 4.2.3, the tropical Kushnirneko–Bernstein theorem (Theorem 3.1.3) takes the following form in terms of these fans.

Theorem 4.2.7. *The intersection number of the hypersurfaces Γ_i in the ring of conditions $\mathscr{R}_n$ is equal to the intersection number of the tropical fans $\mathscr{F}_i$ in the ring $T\mathscr{R}_n$.*

Thus, Theorem 4.2.5 can be regarded as a generalization of the Kushnirenko–Bernstein theorem.

In particular, under the identification of $\mathscr{R}_n$ with $T\mathscr{R}_n$, every hypersurface is sent to the dual tropical fan of the Newton polytope of its equation. Since both rings are generated by their one-dimensional components, this determines the identification uniquely.

However, there is a more explicit description of the tropical fan corresponding to a cycle of arbitrary codimension. It will be given in the next subsection.

4.2.4. *Tropical fans of torus subvarieties.* Associated with every k-dimensional variety $V \in (\mathbb{C} \setminus \{0\})^n$ is its class in the ring of conditions encoded by the tropical fan $\mathbb{T}V$, the tropicalization of V. We describe this fan more explicitly. This has already been done for zero-dimensional cycles and hypersurfaces, and now we do the same for an arbitrary k.

The support of $\mathbb{T}V$ is the k-dimensional skeleton Σ of a complete fan such that the corresponding toric variety X is a good compactification of V, that is, the closure $\overline{V}$ intersects the orbits of codimension k at isolated points. Therefore, for every such orbit σ, its intersection number i_σ with $\overline{V}$ is well defined. Ascribing weight i_σ to the cone of Σ corresponding to σ, we turn Σ into a balanced weighted fan, which is equal to $\mathbb{T}V$.

It turns out (see [62]) that after an appropriate choice of the fan Σ, the closure $\overline{V}$ will have the Cohen–Macaulay property at every point of intersection with the k-dimensional orbit σ of the toric variety X. This simplifies the algebraic calculation of the intersection number i_σ (see, for example, the explicit algebraic formula in [43]).

4.2.5. *Relationship with correspondence theorems.* Let $A \ni 0$ be a finite set of monomials in two variables. We write $\mathbb{C}_1^A$ for the space of all polynomials that are linear combinations of these monomials with constant term 1. Recall that the Severi variety S_d of plane curves with d nodes is the closure in $\mathbb{C}_1^A$ of the set of all polynomials φ such that the equation $\varphi = 0$ determines a reduced irreducible curve in $(\mathbb{C} \setminus \{0\})^2$ with d simple self-intersection points and no other singularities.

Mikhalkin's correspondence theorem can be interpreted as follows in terms of the tropical fan $\mathbb{T}S_d$ of the Severi variety.

The correspondence theorem studies the number of curves in S_d passing through a fixed set of q generic points $z_1, \dots, z_q$ in the plane (where q is the unique number such that the number of such curves is finite and positive).

Let $H_i \subset \mathbb{C}_1^A$ be the incidence hyperplane $\{\varphi \mid \varphi(z_i) = 0\}$. We note that the desired curves correspond to the points of intersection of the variety S_d and the hyperplanes H_i.

Thus, the desired quantity is the intersection number of S_d and the planes H_i. It is equal to the intersection number of their tropical fans $\mathbb{T}S_d$ and $\mathbb{T}H_i$. This number is by definition equal to the weighted number of points of intersection of $\mathbb{T}S_d$ and the tropical hypersurfaces N_i that are shifts of the fans $\mathbb{T}H_i$ by generic vectors.

The tropical hypersurface N_i can be interpreted as an incidence set. Regarding the ambient space $\mathbb{T}_1^A$ as the space of tropical polynomials with support $A \ni 0$ and constant term 1, we write N_i as $\{\varphi \mid \varphi(z_i) = 0\}$, where φ is a tropical polynomial, z_i is a generic point of the tropical plane, and equality to zero is understood as in § 3.1 (that is, z_i lies on the tropical curve defined by the polynomial φ).

Then it turns out that the polynomials φ_j that are points of intersection of $\mathbb{T}S_d$ and the N_j define precisely the nodal tropical curves (in the sense of Mikhalkin's correspondence theorem) passing through the generic points z_i.

Thus, if we know the tropical fan of the Severi variety S_d, then the tropical correspondence theorem for curves with d simple self-intersection points can be deduced from this description. (Note, however, that this tropical fan is fully known only for small d.)

More generally, if A is a finite set of monomials in n variables and $S \subset (\mathbb{C}\setminus\{0\})^A$ is the closure of the set of all polynomials φ such that the hypersurface φ has a prescribed set of singularities, then it suffices to know the tropical fan $\mathbb{T}S$ in order to establish a tropical correspondence theorem for hypersurfaces with a given support set of the equation and a given set of singularities.

In particular, such information is available for the set of all hypersurfaces with one non-trivial singularity, that is, for the Gelfand–Kapranov–Zelevinsky A-discriminant [21]. This made it possible to obtain a tropical correspondence theorem for such hypersurfaces in [45]. In § 3.5 we sketched a proof of this result for $n = 2$.

Subsequently, the tropical fan of the set of hypersurfaces with two non-trivial singularities was described in [16] (see also [12] in the case when $n = 1$). This description is based on the calculus of characteristic classes of subvarieties of a complex torus with values in the ring of conditions. The next subsection is devoted to this calculus.

4.3. Tropical characteristic classes. For brevity, we denote by C_k the component of codimension k in the ring of conditions C of the complex torus $(\mathbb{C}\setminus\{0\})^n$. The existence of the following object was proved in [16].

Definition 4.3.1. A *tropical characteristic class* is a map which sends every algebraic subset $V \subset (\mathbb{C}\setminus\{0\})^n$ to an element $\langle V\rangle = \langle V\rangle_0 + \cdots + \langle V\rangle_n \in C$, $\langle V\rangle_i \in C_i$, and has the following properties:

1) If $V \subset (\mathbb{C}\setminus\{0\})^n$ is of codimension k, then $\langle V\rangle_i = 0$ for $i < k$, $\langle V\rangle_k$ is the class of V in C_k, and $\langle V\rangle_n \in C_0 = \mathbb{Z}$ coincides with the Euler characteristic $e(V)$.

2) For arbitrary U, $V \subset (\mathbb{C}\setminus\{0\})^n$ and a generic $g \in (\mathbb{C}\setminus\{0\})^n$ we have $\langle U \cap gV\rangle = \langle U\rangle\langle V\rangle$.

3) For complex tori X and Y and any algebraic subsets $U \subset X$ and $V \subset Y$ we have $\langle U \times V\rangle = \langle U\rangle \times \langle V\rangle$.

4) The map sending the characteristic function of V to $\langle V\rangle$ extends by linearity to the space of all constructive functions $(\mathbb{C}\setminus\{0\})^n \to \mathbb{Z}$. In particular, $\langle U\cap V\rangle + \langle U\cup V\rangle = \langle U\rangle + \langle V\rangle$. (We recall that a *constructive function* is a linear combination of the characteristic functions of algebraic sets.)

5) Given a morphism $p\colon X \to Y$ of complex tori and an algebraic subset $V \subset X$, we have $p_*\langle V\rangle = \langle p_*V\rangle$. Here $p_*V\colon Y \to \mathbb{Z}$ is the *MacPherson direct image* of V, the value of which at $y \in Y$ is set to be $e\bigl(p^{-1}(y)\cap V\bigr)$.

6) For a smooth toric compactification $X \supset (\mathbb{C}\setminus\{0\})^n$ such that the affine characteristic class $\langle V\rangle$ is contained in the cohomology ring $H^\bullet(X) \subset C$, this class is Poincaré dual to the *Schwartz–MacPherson class* of V in X (see [44] and [52]).

Note that the affine characteristic class is uniquely determined by property 6) as well as by 1)–5). It is also important to stress that in 6) we take the characteristic class of the non-closed (in X) constructive set V rather than of its closure.

We now calculate $\langle V\rangle$ assuming that V is a good variety, that is, one can find a fan Σ such that the closure of V in the corresponding toric compactification $X_\Sigma \supset (\mathbb{C}\setminus\{0\})^n$ is smooth and intersects the orbits in X_Σ transversally.

For every cone $\Gamma \in \Sigma$, we denote by V_Γ the intersection of the closure of V and the Γ-orbit in X_Σ.

Assertion 4.3.2 ([16], see also [23]). *If V is a good variety, then the class $\langle V\rangle_i \in C_i$ is represented by the weighted fan $(P, \varphi), P \subset \mathbb{Q}^n$, $\varphi\colon P \to \mathbb{Q}$, where P is the union of all cones of codimension i in Σ, and the value of φ on each such cone Γ is equal to the Euler characteristic $e(V_\Gamma)$.*

Example 4.3.3. If V is a generic hypersurface with Newton polytope Δ, then a calculation of $e(V_\Gamma)$ by the Kushnirenko formula yields $\langle V\rangle = [\Delta]/(1 + [\Delta])$ or $\langle V\rangle_i = -(-[\Delta])^i$, where $[B]$ is the dual tropical fan of a polytope B (see Definition 3.3.2).

Furthermore, if $V_1, \dots, V_k$ is a k-tuple of generic hypersurfaces with Newton polytopes $\Delta_1, \dots, \Delta_k$, then in view of the multiplicativity of affine characteristic classes, we have

$$\langle V_1 \cap \dots \cap V_k\rangle = \frac{[\Delta_1] \cdots [\Delta_k]}{(1 + [\Delta_1]) \cdots (1 + [\Delta_k])} \,. \tag{4.4}$$

In particular, this yields the formula in [35] for the Euler characteristic of a nondegenerate complete intersection. Furthermore, the right-hand side of (4.4), which appeared as a formal expression in [35] and subsequent publications, becomes geometrically meaningful.

5. Good compactifications of torus subvarieties

We shall show that for every m-dimensional algebraic variety $X \subset (\mathbb{C}^*)^n$ there is a fan W such that the corresponding toric compactification $M_W \supset (\mathbb{C}^*)^n$ possesses the following property: *the closure of X is disjoint from the orbits of M_W of codimension greater than m.* We shall describe all toric compactifications with this property and give a qualitative description of the set of asymptotic values of meromorphic curves lying on X.

Toric varieties form a remarkable class of completions of $(\mathbb{C}^*)^n$. Let X be an algebraic subvariety of the torus $(\mathbb{C}^*)^n$. How can we choose a toric compactification M_W such that the closure $\overline{X} \subset M_W$ of X has the simplest structure?

When X is given by a sufficiently generic system of polynomial equations with fixed Newton polytopes, we can choose a non-singular projective toric compactification M_W in such a way that the variety $\overline{X} \subset M_W$ is non-singular and transversal to all the orbits of M_W (see [34]). There are many compactifications M_W with this property. All of them can be described explicitly in terms of the Newton polytopes of the defining equations of X. This construction enables one to calculate the principal discrete invariants of X explicitly in terms of Newton polytopes. The theory of Newton polytopes relies on this construction to a considerable extent.

When X is a singular variety, $\overline{X}$ is also singular for every toric compactification (since $X = \overline{X} \cap (\mathbb{C}^*)^n$). Even when X is non-singular, there is, generally speaking, no compactification M_W such that $\overline{X}$ is transversal to the orbits of M_W or at least smooth.

However, for every variety X all of whose components are of dimension m, one can choose a toric compactification M_W in such a way that $\overline{X}$ is disjoint from those orbits of M_W whose codimension is greater than m. There are many compactifications M_W with this property, and some of them are smooth projective compactifications. They can all be described explicitly in terms of the Bergman cone of X (see below). This section is devoted to a proof and a thorough discussion of this theorem. We build upon the tropical basis theorem (see, for example, [32]), but make no use of its proof. Below we discuss the statement of this theorem in detail and provide some relevant additional information.

We now present one of the central results in this section. It can be stated without any mention of toric varieties. The leading term of the germ of a meromorphic curve $f\colon (\mathbb{C},0)\to(\mathbb{C}^*)^n$, $f(t)=at^k+\cdots$, is equal to at^k, where $a\in(\mathbb{C}^*)^n$ and $t^k=(t^{k_1},\dots,t^{k_n})$. A ray $l=\{\lambda k\}$, $\lambda\geqslant 0$, is said to be essential for a subvariety $X\subset(\mathbb{C}^*)^n$ if there is a curve germ $f\colon(\mathbb{C},0)\to X$ such that the vector degree k of its leading term is non-zero and lies in l. By Theorem 5.4.6, for a variety X all of whose components are of dimension m there exists a so-called Bergman cone $B(X)$ with the following properties: 1) $B(X)\subset(\mathbb{R}^n)^*$ is a finite union of m-dimensional closed rational cones[1]; 2) a ray l is essential for X if and only if $l\in B(X)$ and l is a rational ray.

Note that this cone coincides with the support of the tropical fan of X, which was defined in § 4. This fact is known as Kapranov's theorem (see, for example, [13]).

The first construction associating a real m-dimensional cone $K(X)$ in $(\mathbb{R}^n)^*$ with every m-dimensional variety $X\subset(\mathbb{C}^*)^n$ appeared in Bergman's pioneering paper [3]. His definition of the cone $K(X)$ was based on analysis instead of algebra; it differs substantially from the definition of $B(X)$, nevertheless $K(X)=B(X)$. The definition of $K(X)$ and the equality $K(X)=B(X)$ are unimportant for our description of the ring of conditions, and we do not consider this in the present paper.

5.1. The tropical basis theorem. In this subsection we recall the theorem on the existence of a tropical basis in any ideal of the ring of Laurent polynomials in n variables [32]. This theorem plays a key role in our paper. We also prove some ramifications of this result.

5.1.1. *The torus, its characters, and Laurent polynomials.* Associated with the torus $(\mathbb{C}^*)^n$ is the character space $\mathbb{R}^n$, which contains the character lattice $\mathbb{Z}^n$. A point $m=(m_1,\dots,m_n)\in\mathbb{Z}^n$ in this lattice can be identified with the character (a monomial) $\chi_m=z^m=z_1^{m_1}\cdots z_n^{m_n}$.

A *Laurent polynomial* $P=\sum c_m z^m$ is a linear combination of the monomials z^m with complex coefficients c_m. Laurent polynomials are regular functions on the torus $(\mathbb{C}^*)^n$, and every rational function $P\colon(\mathbb{C}^*)^n\to\mathbb{C}$ without poles on the torus is a Laurent polynomial. The *support* $S(P)$ *of a Laurent polynomial* $P=\sum c_m z^m$ is the set of points $m\in\mathbb{Z}^n$ such that $c_m\neq 0$. The *Newton polytope* $\Delta(P)$ of a Laurent polynomial P is the convex hull of the support $S(P)$ in the space of characters $\mathbb{R}^n$.

The *truncation* $P^{(\xi)}$ *of order* ξ of a Laurent polynomial $P=\sum c_m z^m$ can be defined for every linear function $\xi\colon\mathbb{R}^n\to\mathbb{R}$ on this space. By definition, $P^{(\xi)}=$

[1] In this section we write $\mathbb{R}^n$ (resp. $(\mathbb{R}^n)^*$) for the character space (resp. the space of one-parameter subgroups) of the torus $(\mathbb{C}^*)^n$.

$\sum_{m\in B} c_m z^m$, where B is the subset of $S(P)$ on which the linear function ξ attains its *minimum*.

We write $\mathscr{R}$ for the ring of Laurent polynomials on $(\mathbb{C}^*)^n$. For every ideal I in $\mathscr{R}$ and every order ξ we have an ideal $I^{(\xi)}$ generated by the truncations of order ξ of all Laurent polynomials in I. (When $\xi = 0$, the ideals I and $I^{(\xi)}$ coincide.)

5.1.2. *Tropical bases of ideals.* A finite set $\{Q_j\} \subset I$ is called a *tropical basis of an ideal* I if, for every order ξ, the ideal $I^{(\xi)}$ is generated by the Laurent polynomials $\{Q_j^{(\xi)}\}$. The ring $\mathscr{R}$ possesses the following tropical Noether property: every ideal I of $\mathscr{R}$ has a tropical basis. Here we state the relatively recent [32] strongest version of this known theorem.

Consider an arbitrary bounded domain U in the character space $\mathbb{R}^n$. A finite set $M \subset I$ is called a U-approximation of the ideal I if, for every $P \in I$ with $\Delta(P) \subset U$ there exists a $Q \in M$ such that $\Delta(Q) = \Delta(P)$. For a fixed domain U, in every ideal I there exists a U-approximation containing at most $N(U)$ elements, where $N(U)$ is the number of distinct lattice polytopes in U. We fix the standard metric in $\mathbb{R}^n$ and write B_ρ for the open ball of radius ρ centred at the origin.

Theorem 5.1.1 (tropical basis theorem). *There exists a function $R = R(r,n)$ such that if $\Delta(P) \subset B_r$ for all elements P in some basis of the ideal I, then every B_R-approximation of I is a tropical basis of I.*

The proof of this theorem uses Seidenberg's theorem (which sharpens the Noether property of the polynomial ring) and the technique of Gröbner bases. The function $R(r,n)$ featuring in the theorem can be described explicitly. But the theorem can have no practical applications since $R(r,n)$ grows too rapidly as a function of n.

5.1.3. *Results supplementary to the tropical basis theorem.* We need some results supplementary to the tropical basis theorem. Let $\bar\xi = (\xi_1,\dots,\xi_k)$ be an ordered tuple of covectors. It defines a map $\pi_{\bar\xi}\colon \mathbb{R}^n \to \mathbb{R}^k$ by $\pi_{\bar\xi}(x) = (\langle\xi_1,x\rangle,\dots,\langle\xi_k,x\rangle) \in \mathbb{R}^k$. For every monomial $m \in \mathbb{Z}^n \subset \mathbb{R}^n$ its $\bar\xi$-degree is $q = \pi_{\bar\xi}(m) \in \mathbb{R}^k$. A Laurent polynomial P is said to be homogeneous of $\bar\xi$-degree q if all the monomials occurring in P are of $\bar\xi$-degree q. We endow the space $\mathbb{R}^k$ with lexicographical ordering (that is, $(x_1,\dots,x_k) > (y_1,\dots,y_k)$ if there is a $p \leqslant k$ such that $x_i = y_i$ for each $i < p$ and $x_p > y_p$).

For every tuple $\bar\xi$ the truncation $P^{(\bar\xi)}$ of multi-order $\bar\xi$ of the Laurent polynomial $P = \sum c_m z^m$ can be defined as $P^{(\bar\xi)} = \sum_{m\in B} c_m z^m$, where B is the subset of the support $S(P)$ of P on which the $\bar\xi$-degree $\pi_{\bar\xi}(m)$ attains its *minimum* value.

Lemma 5.1.2. *Let $A \subset \mathbb{Z}^n$ be a finite set and let $\bar\xi$ be a fixed tuple of covectors. Then there exists a $\xi \in (\mathbb{R}^n)^*$ such that for any $m_1, m_2 \in A$ the inequalities $\pi_{\bar\xi}(m_1) > \pi_{\bar\xi}(m_2)$ and $\langle\xi,m_1\rangle > \langle\xi,m_2\rangle$ are equivalent.*

Proof. Put $M = \max_{\xi_i\in\bar\xi, m\in A} |\langle\xi_i,m\rangle|$, where the ξ_i are the covectors in $\bar\xi$ and $m \in A$. We can easily verify that it suffices to choose ξ to be $\xi_1 + \varepsilon\xi_2 + \dots + \varepsilon^{k-1}\xi_k$, where $0 < \varepsilon < \max\{1, M/(M+1)\}$. $\square$

For every ideal I of the ring $\mathscr{R}$ and every multi-order $\bar\xi$ there is an ideal $I^{(\bar\xi)}$ generated by the truncations of multi-order $\bar\xi$ of all the Laurent polynomials in I.

Theorem 5.1.3. *If $\{Q_j\}$ is a tropical basis of I, then for every multi-order $\bar{\xi}$ the truncations $\{Q_j^{(\bar{\xi})}\}$ form a basis of $I^{(\bar{\xi})}$.*

Proof. We need to prove that, for every $P \in I$, the truncation $P^{(\bar{\xi})}$ lies in the ideal generated by $\{Q_j^{(\bar{\xi})}\}$. Let A be the union of the supports of the Laurent polynomials P and $\{Q_j\}$, and let ξ be the covector from the previous lemma. Then $P^{(\bar{\xi})} = P^{(\xi)}$ and $\{Q_j^{(\bar{\xi})}\} = \{Q_j^{(\xi)}\}$. The required assertion now follows from the definition of a tropical basis. □

Every Laurent polynomial P expands into a sum of homogeneous components with respect to the $\bar{\xi}$-degree: $P = \sum\limits_{q \in \mathbb{R}^k} P_q$, where P_q is a homogeneous Laurent polynomial of $\bar{\xi}$-degree q.

Lemma 5.1.4. *Along with every Laurent polynomial P, the ideal $I^{(\bar{\xi})}$ contains all its homogeneous components P_q with respect to the $\bar{\xi}$-degree. Under the hypotheses of the previous theorem, P_q can be written in the form $P_q = \sum\limits_j Q_j T_j$, where the sums of the $\bar{\xi}$-degrees of the polynomials Q_j and T_j are equal to q.*

Proof. By definition, the ideal $I^{(\bar{\xi})}$ is generated by $\bar{\xi}$-homogeneous Laurent polynomials. The lemma can be proved in exactly the same way as for ideals that are homogeneous with respect to the ordinary degree. □

5.2. The closure of $X \subset (\mathbb{C}^*)^n$ in an affine toric variety. Let $\mathscr{R}$ be the ring of Laurent polynomials, let $\Sigma \subset (\mathbb{R}^n)^*$ be a fan consisting of a cone σ and all its faces, let M_Σ be the corresponding affine toric variety, and let $\mathscr{R}_\Sigma$ be the ring of regular functions on M_Σ. A Laurent polynomial f belongs to $\mathscr{R}_\Sigma$ if and only if the Newton polytope $\Delta(f)$ lies in the cone $C_\Sigma = \{x \in \mathbb{R}^n \mid \langle x, \sigma \rangle \leqslant 0\}$ dual to σ.

Lemma 5.2.1. *If $f \in \mathscr{R}$ and $f^k \in \mathscr{R}_\Sigma$ for some $k > 0$, then $f \in \mathscr{R}_\Sigma$.*

Proof. Let $\Delta(f)$ be the Newton polytope of a Laurent polynomial f. By hypothesis, $\Delta(f^k) = k\Delta(f) \subset C_\Sigma$. It follows that $\Delta(f) \subset C_\Sigma$, that is, $f \in \mathscr{R}_\Sigma$. □

Given an ideal I of the ring $\mathscr{R}$ of Laurent polynomials, we write I_Σ for the ideal $I \cap \mathscr{R}_\Sigma$ of the ring $\mathscr{R}_\Sigma$. The following fact is a corollary of the lemma just proved.

Corollary 5.2.2. *If $f \in \mathscr{R}$ and $f^k \in I_\Sigma$, then f lies in the radical of the ideal $I_\Sigma \subset \mathscr{R}_\Sigma$. If $I \subset \mathscr{R}$ is a radical ideal, then so is $I_\Sigma \subset \mathscr{R}_\Sigma$.*

Let O be an orbit of the lowest dimension in M_Σ. One can identify O with a torus which is a quotient group of $(\mathbb{C}^*)^n$. Let L_Σ be the maximal linear subspace lying in C_Σ. The characters of the torus O can be identified with points of the lattice $\mathbb{Z}^n \cap L_\Sigma$. The ring $\mathscr{R}_{(O)}$ of regular functions on O can be identified with the subring of $\mathscr{R}$ consisting of all Laurent polynomials whose Newton polytopes lie in L_Σ.

Given a function $F \in \mathscr{R}_\Sigma$, we denote the restriction of F to O by $F|_O$. The map $F \to F\big|_O$ is a ring homomorphism of $\mathscr{R}_\Sigma$ onto $\mathscr{R}_{(O)}$. The lattice points $m \in C_\Sigma$ not lying in L_Σ correspond to the characters $\chi_m \in \mathscr{R}_\Sigma$ such that $\chi_m\big|_O \equiv 0$. This can be stated as follows.

Assertion 5.2.3. *If $\Delta(F) \cap L_\Sigma = \varnothing$, then $F\big|_O \equiv 0$.*

Definition 5.2.4. A Laurent polynomial $F \in \mathscr{R}_\Sigma$ is said to be Σ-reduced if the support function of the polytope $\Delta(F)$ vanishes on σ.

The following assertion is obvious.

Assertion 5.2.5. *A Laurent polynomial $f \in \mathscr{R}_\Sigma$ is Σ-reduced if and only if $\Delta(f) \cap L_\Sigma \neq \varnothing$.*

Let $I_{(O,\Sigma)}$ be the image of $I_\Sigma \subset \mathscr{R}_\Sigma$ under the homomorphism sending every function $F \in \mathscr{R}_\Sigma$ to its restriction $F\big|_O$ to the orbit O. Let ξ be any covector in the interior $|\sigma^0|$ of the support of σ. Every element of $I_{(O,\Sigma)}$ is a Laurent polynomial $Q^{(\xi)}$, where Q is a Σ-reduced Laurent polynomial in the ideal $I \subset \mathscr{R}$ and $Q^{(\xi)}$ is the truncation of Q of order ξ.

Theorem 5.2.6. *A set $\overline{X} \cap O$ coincides with the set of zeros of the ideal $I_{(O,\Sigma)} \subset \mathscr{R}_{(O)}$. In other words, $\overline{X} \cap O$ is given in the orbit O by the system of equations $\{Q_\alpha^{(\xi)} = 0\}$, where Q_α ranges over the set of all Σ-reduced Laurent polynomials in $I \subset \mathscr{R}$.*

Proof. The intersection $\overline{X} \cap O$ is given by the system of equations $F\big|_O = 0$, where $F \in I_\Sigma$. $\square$

Remark 5.2.7. The ideal $I_{(O,\Sigma)} \subset \mathscr{R}_{(O)}$ need not be radical even when $I_\Sigma \subset \mathscr{R}_\Sigma$ is; the variety $\overline{X}$ may be tangent to the orbit O.

A Laurent polynomial F is said to be Σ-reducible if the support function $H_{\Delta(F)}$ of its Newton polytope is linear on the support of σ, that is, if $H_{\Delta(F)}(\xi) = \langle \xi, m(F) \rangle$ for every $\xi \in |\sigma|$. The point $m(F) \in \mathbb{Z}^n$ is determined by this equality up to adding any point of the lattice $L_\Sigma \cap \mathbb{Z}^n$. The Laurent polynomial $\widetilde{F} = F x^{-m(F)}$ is called the Σ-reduction of F. (The Σ-reduction makes sense only for Σ-reducible Laurent polynomials and is uniquely determined up to multiplication by χ_k, where $k \in L_\Sigma \cap \mathbb{Z}^n$.)

Suppose that, for some $\xi \in |\sigma^0|$, the ideal $I \subset \mathscr{R}$ has a basis $\{Q_j\}$ consisting of Σ-reducible Laurent polynomials Q_j. We fix a set $\{\widetilde{Q}_j\}$ of Σ-reductions of these Laurent polynomials.

Corollary 5.2.8. *Under these assumptions, $\{\widetilde{Q}_j^{(\xi)}\}$ is a basis of the ideal $I_{(O,\Sigma)} \subset \mathscr{R}_{(O)}$.*

5.3. The tropical basis of an ideal and the closure of its zero locus.

5.3.1. *The tropical basis and affine toric varieties.* Here we continue to use the notation of § 5.2. Assuming that $\{Q_j\}$ is a tropical basis of an ideal $I \subset \mathscr{R}$, we say that the fan Σ of an affine toric variety is convenient for this basis if all $\{Q_j\}$ are Σ-reducible Laurent polynomials. The following assertion can easily be verified.

Assertion 5.3.1. *A fan Σ is convenient for $\{Q_j\}$ if and only if $|\sigma|$ belongs to a cone in the dual fan $\Delta_b^\perp$ of the polytope $\Delta_b = \sum \Delta(Q_j)$.*

We fix some Σ-reductions $\{\widetilde{Q}_j\}$ of all the Laurent polynomials of a tropical basis $\{Q_j\}$ of an ideal $I \subset \mathscr{R}$. The functions $\{\widetilde{Q}_j\}$ are regular on the affine toric variety M_Σ.

Theorem 5.3.2. *The closure $\overline{X} \subset M_\Sigma$ of the zero locus $X \subset (\mathbb{C}^*)^n$ of I can be given in M_Σ by the system of equations $\{\widetilde{Q}_j = 0\}$.*

Proof. The variety M_Σ splits into orbits under the action of $(\mathbb{C}^*)^n$. For every orbit we claim that the set of solutions of the system that belong to this orbit is equal to its intersection with $\overline{X}$. Indeed, this clearly holds for the $(\mathbb{C}^*)^n$-orbit of maximum dimension since the set of solutions of the system in the torus is equal to X. For an orbit O of minimum dimension, this follows from Corollary 5.2.8 since, by the definition of a tropical basis, the Laurent polynomials $\widetilde{Q}_j^{(\xi)}$ generate the ideal $I^{(\xi)}$. Every orbit O_1 of intermediate dimension corresponds to a face γ of σ. Let Γ be the fan consisting of the cone γ and its faces. Then the orbit O_1 is contained in the affine toric variety $M_\Gamma \subset M_\Sigma$ and is an orbit of minimum dimension in this variety. All the Laurent polynomials $\widetilde{Q}_j$ are Γ-reduced. (The support functions of their Newton polytopes vanish on $|\gamma|$ since $|\gamma| \subset |\sigma|$.) The case of the orbit O_1 can now be studied in the same way as that of O. □

The Laurent polynomials $\widetilde{Q}_j$ are Σ-reduced. Hence, for $\xi \in |\sigma|^0$, the $\widetilde{Q}_j^{(\xi)}$ can naturally be identified with Laurent polynomials in $\mathscr{R}_{(O)}$. We denote these Laurent polynomials by T_j.

Theorem 5.3.3. *The functions $\{T_j\} = \{\widetilde{Q}_j^{(\xi)}\}$ form a tropical basis of the ideal $I_{(O,\Sigma)}$ in the ring $\mathscr{R}_{(O)}$.*

Proof. The space $L_\Sigma \subset \mathbb{R}^n$ is orthogonal to the cone $|\sigma|$, and the lattice $\Lambda = L_\Sigma \cap \mathbb{Z}^n$ in this space can naturally be identified with the lattice of characters of the quotient group O of the torus $(\mathbb{C}^*)^n$. Let $\alpha \in L_\Sigma^*$ be an arbitrary covector of L_Σ and let $\widetilde{\alpha} \in (\mathbb{R}^n)^*$ be any covector of $\mathbb{R}^n$ such that $\pi^*(\widetilde{\alpha}) = \alpha$, where $\pi\colon L_\Sigma \to \mathbb{R}^n$ is the natural embedding. Consider the pair of covectors $\bar{\xi} = (\xi, \widetilde{\alpha})$. By Theorem 5.1.3, the functions $\{\widetilde{Q}_j^{(\bar{\xi})}\}$ form a basis of the ideal $I^{(\bar{\xi})}$. The functions $\{\widetilde{Q}_j^{(\bar{\xi})}\}$ are naturally identified with the Laurent polynomials $\{T_j^{(\alpha)}\}$. The functions $\{T_j^{(\alpha)}\}$ form a basis of the ideal $I^{(\bar{\xi})}$ in the ring $\mathscr{R}$ and, therefore, they form a basis of $I_{(O,\Sigma)}^{(\alpha)}$ in $\mathscr{R}_{(O)}$. □

Let $|\gamma| \subset |\sigma|$ be a face of the cone $|\sigma|$ and let O_1 be the corresponding orbit of M_Σ. There are two affine toric varieties related to $|\gamma|$.

The first one is the n-dimensional toric variety M_Γ constructed from the fan Γ containing γ and its faces. The variety M_Γ is a Zariski open subset of M_Σ. It is the complement of the set of orbits whose closures do not contain O_1 (in particular, $O_1 \subset M_\Gamma$).

The other one is the $(n - \dim_{\mathbb{R}} \gamma)$-dimensional variety $\overline{O}_1$, which is the closure of O_1 in M_Σ. One can naturally identify O_1 with an $(n - \dim_{\mathbb{R}} \gamma)$-dimensional torus which is a quotient of $(\mathbb{C}^*)^n$. The variety $\overline{O}_1$ is an affine toric variety with the action of the torus O_1. Let L_Γ be the maximal linear subspace contained in C_Γ. The characters of the torus O_1 can be identified with points in the lattice $\mathbb{Z}^n \cap L_\Gamma$ in the space $L_\Gamma \subset \mathbb{R}^n$ orthogonal to the cone $\gamma \subset (\mathbb{R}^n)^*$. The ring $\mathscr{R}_{(O_1)}$ of regular functions on O_1 can be identified with the subring of $\mathscr{R}$ consisting of the Laurent polynomials whose Newton polytopes lie in L_Γ.

Let $I \subset \mathscr{R}$ be an ideal in the ring $\mathscr{R}$ of Laurent polynomials, let $\{Q_j\}$ be a tropical basis of I, let X be the zero locus of I in the torus $(\mathbb{C}^*)^n$, and let $\overline{X}$ be the closure of X in M_Σ. We denote the intersections of $\overline{X}$ with the orbits O and O_1 by $X_0 = \overline{X} \cap O$ and $X_1 = \overline{X} \cap O_1$.

Theorem 5.3.4. *If all elements of the tropical basis $\{Q_j\}$ are Σ-reducible Laurent polynomials, then the set of limit points of X_1 in the orbit O coincides with X_0.*

Proof. We fix some Σ-reductions $\{\widetilde{Q}_j\}$ of the elements of the tropical basis $\{Q_j\}$. The variety $\overline{X}$ is given by the system of equations $\widetilde{Q}_j = 0$ in M_Σ (see Theorem 5.3.2). The restrictions of these equations to the orbits O and O_1 determine X_0 and X_1.

Consider the affine subvariety $M_\Gamma \subset M_\Sigma$ with O_1 as its minimal orbit. Let $I_{(O_1,\Gamma)}$ be the image of $I_\Gamma = I \cap \mathscr{R}_\Gamma$ under the restriction of all functions in $\mathscr{R}_\Gamma$ to O_1. Let $\xi \in |\gamma^0|$ be a covector. By Theorem 5.3.3, the functions $\{T_j\} = \{\widetilde{Q}_j^{(\xi)}\}$ form a tropical basis of $I_{(O_1,\Gamma)}$.

We arrive at the lower-dimensional situation we have already studied. In the torus O_1, we have the zero locus X_1 of the ideal $I_{(O_1,\Gamma)}$ and the Laurent polynomials $\{T_j\}$, which form a tropical basis of $I_{(O_1,\Gamma)}$. The closure $\overline{O}_1$ of O_1 in M_Σ is a toric variety with the action of the torus O_1. The cone of $\overline{O}_1$ in the space of characters is $C_\Sigma \cap L_\Gamma \supset L_\Sigma$. The Newton polytopes of the Laurent polynomials $\{T_j\}$ have non-empty intersections with L_Σ and therefore with L_Γ. Hence the toric compactification $\overline{O}_1$ of O_1 is convenient for the tropical basis $\{T_j\}$. By Theorem 5.3.2, $\overline{X}_1 \cap O$ is given by the system of equations $\{T_j\big|_O = 0\}$ on O. By the same theorem, $X_0 \subset O$ is given by the system of equations $\{\widetilde{Q}_j\big|_O = 0\}$ on O. These two systems of equations coincide. Hence $X_0 = \overline{X}_1 \cap O$. □

5.3.2. *Tropical basis and general toric varieties.* Fix an algebraic variety $Y_i \subset O_i$ in each orbit O_i of a toric variety M_W.

Definition 5.3.5. We say that the set of subvarieties $Y_i \subset O_i$ is compatible with respect to closure if, for every pair of orbits O_j and O_i such that O_j lies in the closure of O_i, the variety Y_j coincides with the set of limit points of Y_i in O_j.

We now extend Theorem 5.3.4 to toric varieties M_W that are not affine. Let $X \subset (\mathbb{C}^*)^n$ be the zero locus of an ideal I and let $\{Q_j\}$ be a tropical basis of I. We say that a fan W of the toric variety M_W is convenient for this basis if all the elements of $\{Q_j\}$ are Σ-reducible Laurent polynomials for every affine subfan Σ of W.

Let $\overline{X}$ be the closure of X in M_W. We write $X_i = O_i \cap \overline{X}$ for the intersections of $\overline{X}$ with the orbits O_i of M_W.

Theorem 5.3.6. *If the fan W is convenient for some tropical basis of I, then the varieties $X_i \subset O_i$ are compatible with respect to closure.*

Proof. If O_j lies in the closure of O_i and σ_j, σ_i are the cones corresponding to these orbits, then σ_i is a face of σ_j. The orbits O_i and O_j lie in the affine toric variety M_Σ

whose fan Σ consists of the cone σ_j and all its faces. The desired assertion now reduces to Theorem 5.3.4. $\square$

Theorem 5.3.7. *Suppose that the hypotheses of Theorem 5.3.6 hold. Then*

1) *the union U of all the orbits O of M_W such that $\overline{X} \cap O \neq \varnothing$ is a toric subvariety $M_{W(X)}$ of M_W;*

2) *if all the components of X are of dimension m and $O \subset M_{W(X)}$ is an orbit of codimension k, then every component of $\overline{X} \cap O$ has dimension $m-k$.*

Proof. 1) Suppose that an orbit O_j lies in the closure of an orbit O_i. Then, by Theorem 5.3.6, $X_j = \overline{X} \cap O_j$ is contained in the closure of $X_i = \overline{X} \cap O_i$. Therefore, if $X_i = \varnothing$, then also $X_j = \varnothing$. Hence U is Zariski open and $M_{W(X)}$ is a toric variety.

2) If $O_i \subset M_{W(X)}$ is an orbit of codimension 1, then it is adjacent to the orbit $O_0 = (\mathbb{C}^*)^n$, that is, O_i is a smooth hypersurface in the toric variety $O_i \cup O_0$. The variety X_i consists of the limit points of X in the hypersurface. Hence all the components of X_i are of dimension $m-1$. Suppose that the theorem has been proved for orbits of codimension $p-1$. If $O_j \subset M_{W(X)}$ is an orbit of codimension p, then it is adjacent to some orbit O_i of codimension $p-1$. In this case O_j is a smooth hypersurface in the toric variety $O_j \cup O_i$. The variety X_j consists of the limit points of X_i in the hypersurface. Hence all the components of X_j are of dimension $m-(p-1)-1 = m-p$. $\square$

5.4. The Bergman cone of a variety $X \subset (\mathbb{C}^*)^n$.

5.4.1. *A non-invariant definition of the Bergman cone.* Let X be a subvariety of $(\mathbb{C}^*)^n$ all of whose irreducible components are of dimension m. In Theorem 5.3.7 we introduced the toric variety $M_{W(X)}$. It is not invariantly defined and depends on the choice of an ideal I whose zero locus is X (it is only the radical of I that is invariantly defined), on the choice of a tropical basis of I, and on the choice a convenient fan W for this basis. We define the Bergman cone of X using the fan $W(X)$. In what follows we shall show that the Bergman cone is an invariant of X and is independent of the arbitrary choices of the objects used in its definition.

Definition 5.4.1. The Bergman cone $B(X) \subset (\mathbb{R}^n)^*$ of a variety $X \subset (\mathbb{C}^*)^n$ is the support $|W(X)| \subset (\mathbb{R}^n)^*$ of the fan $W(X)$ of the toric variety introduced in Theorem 5.3.7.

The following assertion follows automatically from the definition of the Bergman cone.

Assertion 5.4.2. *The Bergman cone of a variety all of whose components are of (complex) dimension m is a finite union of closed m-dimensional rational cones.*

5.4.2. *The asymptotic behaviour of curves lying on a variety.* Consider the germ at $0 \in \mathbb{C}$ of a meromorphic map $f\colon (\mathbb{C},0) \to X \subset (\mathbb{C}^*)^n$ from the complex line to X. Let $z_1,\dots,z_n$ be coordinates in $(\mathbb{C}^*)^n$ and t a coordinate in $\mathbb{C}$. Then, up to lower-order terms, f can be written in the form $f = (a_1t^{k_1}+\cdots,\dots,a_nt^{k_n}+\cdots)$ or $f = at^k+\cdots$, where $a=(a_1,\dots,a_n) \in (\mathbb{C}^*)^n$ and $t^k = (t^{k_1},\dots,t^{k_n})$. *Which asymptotic patterns $f(t) = at^k + \cdots$ can occur for curves $f\colon (\mathbb{C},0) \to X \subset (\mathbb{C}^*)^n$ on X?*

The following two assertions can easily be proved.

Assertion 5.4.3. *For every curve $f\colon (\mathbb{C},0)\to X$ with asymptotic behaviour $f(t)=at^k+\cdots$, the coefficient $a\in(\mathbb{C}^*)^n$ is a zero of the ideal $I^{(k)}$.*

Proof. The identity $P(f(t))\equiv 0$ holds for every $P\in I$. But $P(f(t))=P^{(k)}(a)t^m+\cdots$, where $m=H(\Delta(P))(k)$. Hence $P^{(k)}(a)=0$. □

Assertion 5.4.4. *If $a\in(\mathbb{C}^*)^n$ is a zero of the ideal $I^{(k)}$, then there is a curve $f\colon(\mathbb{C},0)\to X$ with asymptotic behaviour $f(t)=at^{kq}+\cdots$, where q is a positive number.*

Proof. When $k=0$, one can take the constant map $f(t)\equiv a$ for f, where $a\in X$ is a zero of the ideal I. When $k\neq 0$, we consider the one-dimensional cone σ in $(\mathbb{R}^n)^*$ generated by the covector k. Let M_Σ be a toric variety whose fan Σ contains σ and the vertex 0. Let Σ be a convenient fan for a tropical basis $\{P_i\}$ of the ideal I. Then any $P\in\{P_i\}$ is Σ-reducible for Σ. Let $\widetilde{P}$ be a Σ-reduction of P. The intersection of $\overline{X}$ with an $(n-1)$-dimensional orbit $O\subset M_\Sigma$ is given by the system of equations $\widetilde{P}^{(k)}=0$ with $P\in I$. Applying an automorphism of $(\mathbb{C}^*)^n$ if necessary, we can assume that σ is the ray $(x_1,0,\dots,0)$, where $x_1\geqslant 0$, and the $(n-1)$-dimensional orbit O is given by the equation $z_1=0$ in $M_\Sigma=\mathbb{C}\times(\mathbb{C}^*)^{n-1}$, where z_1 is the coordinate in $\mathbb{C}$. In such coordinates, the Σ-reduction of a Laurent polynomial P is a Laurent polynomial $\widetilde{P}$ all of whose monomials have non-negative degrees with respect to z_1, and $\widetilde{P}^{(k)}$ is the sum of monomials in P whose degree with respect to z_1 is equal to zero (the set of such monomials in $\widetilde{P}$ must be non-empty). Put $X_0=\overline{X}\cap O$. The zero locus of the ideal $I^{(k)}$ in $(\mathbb{C}^*)^n=\mathbb{C}^*\times(\mathbb{C}^*)^{n-1}$ has the form $\mathbb{C}^*\times X_0$. Since X_0 is the closure of X in O, for every point $b\in X_0$ there is a curve germ $f\colon(\mathbb{C},0)\to X$ such that $\lim_{t\to 0}f(t)=b$. The leading term of the asymptotic formula for this curve has the form ct^m:

$$f=(ct^m+\cdots,b+\cdots),\quad\text{where}\quad m\geqslant 0,\quad c\neq 0.$$

The change of parameter $\tau=dt$ enables us to make the coefficient of τ^m equal to any non-zero number w. Thus we have constructed a curve in X whose vector degree $(m,0,\dots,0)$ is proportional to k and whose coefficient (w,b) is a prescribed zero of the ideal $I^{(k)}$ in the torus $\mathbb{C}^*\times(\mathbb{C}^*)^{n-1}=(\mathbb{C}^*)^n$. □

A covector $k\in(\mathbb{Z}^n)^*\subset(\mathbb{R}^n)^*$ is essential for a variety $X\subset(\mathbb{C}^*)^n$ if there is a curve $f\colon(\mathbb{C},0)\to X$ with asymptotic behaviour $f(t)=at^m+\cdots$ whose vector degree m is equal to kq, where $q>0$.

Corollary 5.4.5. *A covector $k\in(\mathbb{Z}^n)^*$ is essential for the zero locus $X\subset(\mathbb{C}^*)^n$ of an ideal $I\subset\mathscr{R}$ if and only if the truncation $I^{(k)}$ of I in the direction of k does not coincide with the ring $\mathscr{R}$.*

The zero covector is essential for every non-empty variety X. Two covectors differing only by a positive factor are both essential or both not essential for X at the same time. Recall that a rational ray $l\in(\mathbb{Z}^n)^*\subset(\mathbb{R}^n)^*$ is said to be essential for a variety $X\subset(\mathbb{C}^*)^n$ if the integer covectors on l are essential for X.

5.4.3. *The Bergman cone and essential rays.* The following theorem describes the structure of the set of essential rays for a variety X all of whose components are of the same dimension m.

Theorem 5.4.6. *Suppose that all the components of a variety $X \subset (\mathbb{C}^*)^n$ are of dimension m. A rational ray $l \in (\mathbb{R}^n)^*$ is essential for X if and only if l belongs to the Bergman cone $B(X)$ of X. The cone $B(X)$ is equal to the closure of the union of all essential rays for X.*

Proof. Each essential ray l for X is contained in some cone of the fan $W(X)$. Indeed, if a curve $f(t) = at^k + \cdots$ with $0 \neq k \in l$ lies in X, then $z = \lim_{t\to 0} f(t) \in \overline{x}$, and therefore z lies in some orbit $O \subset M_{W(X)}$. Then $k \in |\sigma^0|$, where σ is the cone in $W(X)$ corresponding to O.

Conversely, suppose that $0 \neq k \in |\sigma^0|$, where σ is a cone in $W(X)$. Let Σ be the fan consisting of σ and its faces. All elements $\{Q_j\}$ of a tropical basis of I are Σ-reducible Laurent polynomials. The ideal $I^{(k)}$ is generated by the Laurent polynomials $\{Q_j^{(k)}\}$, which coincide for all $k \in |\sigma^0|$. The zero locus of $I^{(k)}$ is non-empty since the intersection of $\overline{X}$ and O is non-empty. By Assertion 5.4.4, there exists a curve $f(t) = at^{kq} + \cdots$ on X, where a is a zero of the ideal $I^{(k)}$ and $q > 0$. The ray λk, where $\lambda \geqslant 0$, is essential for X. □

Theorem 5.4.6 shows that the Bergman cone is invariantly defined. It also enables us to extend the definition of the Bergman cone to subvarieties $X \subset (\mathbb{C}^*)^n$ with components which may have differing dimensions.

Definition 5.4.7. The Bergman cone $B(X) \subset (\mathbb{R}^n)^*$ of a variety $X \subset (\mathbb{C}^*)^n$ is the closure of the union of the essential rays for X.

Corollary 5.4.8. 1) $B(X \cup Y) = B(X) \cup (Y)$.

2) $B(X \cap Y) \subset B(X) \cap (Y)$.

3) $B(X)$ *is a finite union of rational cones (of differing dimensions).*

4) *A rational ray l is essential for X if and only if $l \subset B(X)$.*

5) *For a variety all of whose components are of dimension m, Definitions 5.4.1 and 5.4.7 are equivalent.*

Proof. Parts 1) and 2) are obvious. Parts 3) and 4) follow from Theorem 5.4.6 and part 1). Part 5) follows from Theorem 5.4.6. □

Let $X \subset (\mathbb{C}^*)^n$ be an algebraic variety, $M_W \supset (\mathbb{C}^*)^n$ a toric variety, and $\overline{X}$ the closure of X in M_W.

Theorem 5.4.9. *The variety $\overline{X}$ is complete if and only if $B(X) \subset |W|$.*

We shall not prove this theorem (it can be proved in the same way as Theorem 5.4.6). To state the following result we need to define the m-dimensional skeleton W_m of a fan W. Namely, W_m is the subfan of W which contains all the cones in W of dimension less than or equal to m.

Let $X \subset (\mathbb{C}^*)^n$ be an algebraic variety all of whose components are of dimension m, let $M_W \supset (\mathbb{C}^*)^n$ be a toric variety, and let $\overline{X}$ be the closure of X in M_W.

Theorem 5.4.10. *The variety $\overline{X}$ is complete and disjoint from the orbits of M_W of codimension greater than m if and only if $B(X) \subset |W_m|$.*

Proof. If $\overline{X}$ is complete, then $B(X) \subset W$ by Theorem 5.4.9. Suppose that $B(X)$ intersects $|\sigma^0|$ for some cone σ in W with $\dim_{\mathbb{R}} \sigma > m$. Then $\overline{X}$ intersects the orbit $O \subset M_W$ corresponding to the cone σ, and the codimension of this orbit is greater than m. Therefore, if $\overline{X}$ is disjoint from orbits of codimension $> m$, then $B(X) \subset |W_m|$.

Conversely, suppose that $B(X) \subset |W_m|$. Then $\overline{X} \subset M_{W_m} \subset M_W$. But M_{W_m} contains only orbits of codimension not greater than m. $\square$

6. Rings with Gorenstein duality

6.1. Gorenstein duality. Let A be a commutative associative unital algebra over a field $\mathbf{k}$. For each $\mathbf{k}$-linear function $L\colon A \to \mathbf{k}$ there is an associated bilinear form B_L on A defined by $B_L(a,b) = L(ab)$. If B_L is a non-degenerate form, then we say that it defines *Gorenstein duality* on A. Each algebra is determined by its generators and relations between these. If a function L defines Gorenstein duality on an algebra, then the latter can be recovered uniquely from the values of L on each polynomial of the generators of the algebra. In this subsection we discuss the properties of algebras endowed with Gorenstein duality and present two slightly different ways to describe such algebras.

The cohomology rings of smooth complete n-dimensional toric varieties and the ring of conditions $\mathscr{R}_n$ of the complex torus $(\mathbb{C}^*)^n$ possess Gorenstein duality. Furthermore, these rings are generated by algebraic hypersurfaces in the corresponding varieties. For this reason, the cohomology rings of toric varieties and the ring $\mathscr{R}_n$ can be recovered from the intersection numbers of all possible systems of n algebraic hypersurfaces in the corresponding varieties. Due to the Kushnirenko–Bernstein theorem, all these intersection numbers can be calculated in terms of the volumes of the Newton polytopes of the defining equations of the hypersurfaces. Hence, these rings can be described it terms of the volumes of convex lattice polytopes in $\mathbb{R}^n$. At the end of this section (see § 6.3.2) we present two such descriptions, which differ slightly from one another.

6.1.1. *Algebras with Gorenstein duality.* Let M_1 and M_2 be two (possibly infinite-dimensional) linear spaces over the field $\mathbf{k}$. A bilinear form B on $M_1 \times M_2$ is called a pairing of these spaces. Using a pairing, to each subspace $M \subset M_1$ we can assign the *orthogonal space* $M^\perp \subset M_2$ of all vectors $b \in M_2$ such that $B(a,b) = 0$ for each $a \in M$.

A pairing is said to be non-degenerate if B is a non-degenerate form, that is, if 1) for each $a \in M_1 \setminus \{0\}$ there exists a $b \in M_2$ such that $B(a,b) \neq 0$, and 2) for each $b \in M_2 \setminus \{0\}$ there exists an $a \in M_1$ such that $B(a,b) \neq 0$.

A non-degenerate pairing defines an embedding of M_2 into the dual space M_1^* of M_1, which assigns to each $b \in M_2$ the linear function l_b on M_1 for which $l_b(a) = B(a,b)$. If M_1 is a finite-dimensional space, then the embedding establishes an isomorphism between M_2 and M_1^*; in particular, $\dim M_1 = \dim M_2$.

Let $\mathscr{A}$ be a commutative associative unital algebra over $\mathbf{k}$.

Definition 6.1.1. For each $\mathbf{k}$-linear function $\mathscr{L}\colon \mathscr{A} \to \mathbf{k}$ there is an associated *symmetric* $\mathbf{k}$*-bilinear form* $B_{\mathscr{L}}$ on $\mathscr{A}$ defined by

$$B_{\mathscr{L}}(a,b) = \mathscr{L}(ab) \in \mathbf{k}.$$

The linear function $\mathscr{L}$ on $\mathscr{A}$ defines *Gorenstein duality* if $B_{\mathscr{L}}$ is a non-degenerate form.

Example 6.1.2. Let A be a subspace of $L^2(\mathbb{R}^n, d\mu)$ that is closed under multiplication of functions. Let L be the functional that assigns to each function $f \in A$ the integral $\displaystyle\int_{\mathbb{R}^n} f\phi\, d\mu$, where $\phi \in A$ is a weight function such that the set $\phi^{-1}(0)$ has measure zero. Then L defines Gorenstein duality on A. Indeed, if $f \neq 0$, then $B_L(f, \phi f) = \displaystyle\int_{\mathbb{R}^n} f^2\phi^2\, d\mu > 0$.

Example 6.1.3. Let A be the $\mathbb{C}$-algebra of complex-valued functions on a finite set X. Let L be the functional that assigns to a function $f \in A$ the complex number $\displaystyle\sum_{x\in X} f(x)\phi(x)$, where $\phi \in A$ is a weight function none of whose values is 0. The functional L defines Gorenstein duality on A (because $B_L(f, \bar{\phi}\bar{f}) = \displaystyle\sum_{x\in X} |f(x)|^2|\phi(x)|^2 > 0$ if $f \neq 0$).

Example 6.1.4. Let M be a compact oriented real manifold of even dimension $2n$ and with no boundary. Let A be the commutative subalgebra of the cohomology ring $H^*(M, \mathbb{R})$, which consists of linear combinations of even-dimensional elements, that is, $\alpha \in A$ if

$$\alpha = \sum_{0\leqslant k\leqslant n} \alpha_{2k}, \tag{6.1}$$

where $\alpha_{2k} \in H^{2k}(M, \mathbb{R})$. Let L be the functional that sends $\alpha \in A$ given by (6.1) to the value of the cohomology class α_{2n} on the fundamental cycle of M. By Poincaré duality, B_L is a non-degenerate form and L defines Gorenstein duality on A.

Lemma 6.1.5. *Let $B_{\mathscr{L}}$ ba a (possibly degenerate) form on $\mathscr{A}$. Then*

1) *the orthogonal space $J^{\perp_{\mathscr{L}}} \subset \mathscr{A}$ of an ideal $J \subset \mathscr{A}$ with respect to $B_{\mathscr{L}}$ is an ideal of $\mathscr{A}$;*

2) *the kernel* $\ker B_{\mathscr{L}}$ *of $B_{\mathscr{L}}$ is an ideal of $\mathscr{A}$.*

Proof. 1) Let $b \in J^{\perp_{\mathscr{L}}}$ and $c \in \mathscr{A}$. Then for each $a \in J$ we have $B_{\mathscr{L}}(bc, a) = \mathscr{L}(bca) = 0$ because $ca \in J$. Hence $J^{\perp_{\mathscr{L}}}$ is an ideal of $\mathscr{A}$.

2) The kernel $\ker B_{\mathscr{L}}$ is orthogonal to $\mathscr{A}$ with respect to $B_{\mathscr{L}}$. Hence part 2) is a consequence of 1). □

The annihilator J_{an} of an ideal $J \subset A$ is the ideal consisting of all $a \in A$ such that $ab = 0$ for all $b \in J$.

Lemma 6.1.6. *If B_L is a non-degenerate form, then $J^{\perp_L} = J_{\mathrm{an}}$ for each ideal $J \subset A$.*

Proof. The inclusion $J_{\mathrm{an}} \subset J^{\perp_L}$ is clearly true even when B_L is degenerate. Conversely, suppose that for some $a \in J^{\perp_L}$ there exists a $b \in J$ such that $ab \neq 0$. Then, since B_L is non-degenerate, there exists a $c \in A$ such that $L((ab)c) \neq 0$. Hence $L(a(bc)) \neq 0$ and a is not orthogonal to $bc \in J$. This contradiction shows that $J_{\mathrm{an}} \supset J^{\perp_L}$, and therefore $J_{\mathrm{an}} = J^{\perp_L}$. □

Corollary 6.1.7. *If L defines Gorenstein duality on A, then for each ideal $J \subset A$ the ideal $J^{\perp_L}$ is independent of the choice of L.*

Corollary 6.1.8. *If Gorenstein duality exists on A, then*

1) *for each ideal J of finite codimension $m = \dim A/J$ the ideal J_{an} is finite-dimensional and $J_{\mathrm{an}} = m$; furthermore, $(J_{\mathrm{an}})_{\mathrm{an}} = J$;*

2) *for each ideal J of finite dimension $m = \dim J$ the ideal J_{an} has finite codimension and $\dim A/J_{\mathrm{an}} = m$; furthermore, $(J_{\mathrm{an}})_{\mathrm{an}} = J$.*

Proof. If the form B_L is non-degenerate, then it induces an isomorphism of the spaces $(A/J)^*$ and J under the assumptions in 1). Under the assumptions in 2), it induces an isomorphism of the spaces J and $(A/J)^*$. □

This corollary establishes necessary conditions for Gorenstein duality to exist on an algebra A. In the next subsection we show that for local algebras these conditions are sufficient.

6.1.2. *Local algebras with Gorenstein duality.* In this subsection we consider commutative associative unital $\mathbf{k}$-algebras $\mathscr{A}$ with the following properties:

(a) $\mathscr{A}$ is a local algebra, that is, $\mathscr{A}$ contains a unique maximal ideal $\mathbf{m}$;

(b) the residue field $\mathscr{A}/\mathbf{m}$ is isomorphic to the ground field $\mathbf{k}$;

(c) there exists a positive integer k such that $\mathbf{m}^k = 0$.

Let $\mathbf{m}_{\mathrm{an}} \subset \mathscr{A}$ be the annihilator of the maximal ideal $\mathbf{m}$ of an algebra $\mathscr{A}$ with properties (a)–(c).

Lemma 6.1.9. 1) *Each non-trivial ideal $I \subset A$ contains a non-zero element m of $\mathbf{m}_{\mathrm{an}}$.*

2) *The ideal (m) generated by an arbitrary non-zero element $m \in \mathbf{m}_{\mathrm{an}}$ is a one-dimensional space over $\mathbf{k}$.*

Proof. 1) Let l be the largest integer such that the ideal $I \cdot \mathbf{m}^l$ is non-trivial. Each non-zero element m of $I \cdot \mathbf{m}^l$ lies in $\mathbf{m}_{\mathrm{an}}$ because $m \cdot \mathbf{m} \subset I \cdot \mathbf{m}^{l+1} = 0$.

2) Since $m \cdot \mathbf{m} = 0$, it follows that $m \cdot \mathscr{A} = m \cdot (\mathscr{A}/\mathbf{m}) = m \cdot \mathbf{k}$. □

Lemma 6.1.10. *The bilinear form $B_{\mathscr{L}}$ on an algebra $\mathscr{A}$ with properties* (a)–(c) *is non-degenerate if and only if $\mathscr{L}$ does not vanish at any non-zero element m of $\mathbf{m}_{\mathrm{an}}$. Gorenstein duality can be defined on $\mathscr{A}$ if and only if $\mathbf{m}_{\mathrm{an}}$ is a one-dimensional space over $\mathbf{k}$.*

Proof. Suppose that $\mathscr{L}$ does not vanish at any non-zero elements of $\mathbf{m}_{\mathrm{an}}$. Applying part 1) of Lemma 6.1.9 to the principal ideal $(a) = I$ generated by a non-zero element $a \in \mathscr{A}$, we can see that there exists a $b \in \mathscr{A}$ such that ab is non-zero and $ab \in \mathbf{m}_{\mathrm{an}}$. Hence the form $B_{\mathscr{L}}$ is non-degenerate.

If $\mathscr{L}$ vanishes at a non-zero element $a \in \mathbf{m}_{\mathrm{an}}$, then a lies in the kernel of $B_{\mathscr{L}}$, and therefore the form $B_{\mathscr{L}}$ is degenerate. If the $\mathbf{k}$-linear space $\mathbf{m}_{\mathrm{an}}$ has dimension greater than 1, then every linear function $\mathscr{L}$ vanishes at some non-zero element $a \in \mathbf{m}_{\mathrm{an}}$. Hence all the bilinear forms $B_{\mathscr{L}}$ on $\mathscr{A}$ are degenerate.

On the other hand, if the space $\mathbf{m}_{\mathrm{an}}$ is one-dimensional and $\mathscr{L}$ is not identically zero on $\mathbf{m}_{\mathrm{an}}$, then the form $B_{\mathscr{L}}$ is non-degenerate. □

6.1.3. *Quotient algebras with Gorenstein duality.* In this subsection we describe all quotient algebras of a commutative associative unital algebra $\mathscr{A}$ that can be endowed with Gorenstein duality.

Consider the action ρ of $\mathscr{A}$ on the dual space $\mathscr{A}^*$ induced by multiplication in $\mathscr{A}$: for a linear function $\mathscr{L} \in \mathscr{A}^*$ and an element $w \in \mathscr{A}$ the function $\rho(w)\mathscr{L}$ is defined by $\rho(w)\mathscr{L}(v) = \mathscr{L}(wv)$. With every linear function $\mathscr{L} \in \mathscr{A}^*$ we associate the set $J_{\mathscr{L}} \subset A$ of all elements w such that $\mathscr{L}_w = \rho(w)\mathscr{L}$ is identically equal to zero. By definition, $J_{\mathscr{L}}$ coincides with the kernel of $B_{\mathscr{L}}$. By Lemma 6.1.5, $J_{\mathscr{L}} \subset \mathscr{A}$ is an ideal of $\mathscr{A}$.

Theorem 6.1.11. 1) *The function $L\colon \mathscr{A}_{\mathscr{L}} \to \mathbf{k}$ induced by $\mathscr{L}$ on the quotient algebra $\mathscr{A}_{\mathscr{L}} = \mathscr{A}/J_{\mathscr{L}}$ defines Gorenstein duality.*

2) *Conversely, assume that a $\mathbf{k}$-linear function L on the quotient algebra $A = \mathscr{A}/J$ defines Gorenstein duality. Then $J = J_{\mathscr{L}}$, where $\mathscr{L} = \pi^*L$ with $\pi\colon \mathscr{A} \to \mathscr{A}/J$ being the factorization homomorphism.*

Proof. 1) The function $\mathscr{L}$ vanishes on $J_{\mathscr{L}}$ because $\mathscr{L}(w) = \mathscr{L}_w(e) = 0$ if $w \in J_{\mathscr{L}}$. Hence $\mathscr{L}$ induces a function L on $\mathscr{A}/J_{\mathscr{L}}$. The kernel of B_L is trivial, for if $x \in \ker B_L$, then $w = \pi^{-1}(x) \in \ker B_{\mathscr{L}}$, where $\pi\colon \mathscr{A} \to \mathscr{A}/J$ is the factorization homomorphism. Therefore, $w \in J_{\mathscr{L}}$ and $x = \pi(w) = 0$.

2) If the form B_L on $\mathscr{A}/J$ is non-degenerate, then the kernel of $B_{\mathscr{L}}$ on $\mathscr{A}$, where $\mathscr{L} = \pi^*L$, coincides with $J \subset \mathscr{A}$. □

Using this theorem, we can find quotient algebras of $\mathscr{A}$ endowed with Gorenstein duality with the help of a non-zero linear function $\mathscr{L}$ on $\mathscr{A}$.

Example 6.1.12. Let $\mathscr{A} = \mathbb{C}\{\{x_1, \dots, x_n\}\}$ be the ring of analytic function germs at $0 \in \mathbb{C}^n$. Suppose that $f_1, \dots, f_n \in \mathscr{A}$ and 0 is an isolated root of the system of equations

$$f_1 = \cdots = f_n = 0. \tag{6.2}$$

On $\mathscr{A} = \mathbb{C}\{\{x_1, \dots, x_n\}\}$ we can define a linear functional $\mathscr{L}$ whose value at a germ $h \in \mathscr{A}$ is given by

$$\mathscr{L}(h) = \frac{1}{(2\pi)^n} \int_{|f_1| = \cdots = |f_n| = \varepsilon} \frac{h(x)\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n}{f_1 \cdots f_n},$$

where ε is a sufficiently small positive number. The functional $\mathscr{L}$ on $\mathscr{A}$ defines the quotient algebra $A_{\mathscr{L}}$ possessing Gorenstein duality. We can show that the *quotient algebra $\mathscr{A}_{\mathscr{L}}$ coincides with the finite-dimensional local $\mathbb{C}$-algebra*

$$A = \mathscr{A}/(f_1, \dots, f_n)$$

connected with the 0 root of the system (6.2). The minimal ideal $\mathbf{m}_{\mathrm{an}}$ of A is generated by the germ of the Jacobian

$$J(x) = \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)}$$

of the system (6.2) at 0. The value of $\mathscr{L}$ at J is the multiplicity μ of the 0 root of (6.2). A linear function $L\colon A \to \mathbb{C}$ defies a non-degenerate form B_L on A if and only if $L(J) \neq 0$.

Example 6.1.13. The above example also has a real version. Let $\mathscr{A} = \mathbb{R}\{\{x_1, \dots, x_n\}\}$ be the ring of germs of real analytic functions at $0 \in \mathbb{R}^n$. Let $f_1, \dots, f_n \in \mathscr{A}$, and assume that the local $\mathbb{R}$-algebra

$$A = \mathbb{R}\{\{x_1, \dots, x_n\}\}/(f_1, \dots, f_n)$$

is finite-dimensional.

Gorenstein duality exists on A. The minimal ideal $\mathbf{m}_{\mathrm{an}}$ of A is generated by the germ at 0 of the Jacobian J of the system $f_1 = \dots = f_n = 0$. A linear function $L\colon A \to \mathbb{R}$ defines a non-degenerate real form B_L on A if and only if $L(J) \neq 0$.

Consider the germ of the vector field $V = (f_1, \dots, f_n)$ at $0 \in \mathbb{R}^n$. The classical Eisenbud–Levine formula gives the index of the vector field V at 0. Namely, the *index of V is equal to the signature of the quadratic form K on A defined by $K(f) = L(f^2)$, where L is an arbitrary real $\mathbb{R}$-linear function on A such that $L(J) > 0$.*

6.1.4. *The symmetric algebra of a linear space.* With a $\mathbf{k}$-linear space M we associate its *symmetric algebra* $\mathscr{S}(M)$, which is the free associative commutative unital $\mathbf{k}$-algebra generated by the vectors in M. The elements of $\mathscr{S}(M)$ can be regarded as formal polynomials in vectors from M with coefficients in $\mathbf{k}$. This gives a graded algebra

$$\mathscr{S}(M) = \mathscr{A}_0 + \mathscr{A}_1 + \dots + \mathscr{A}_k + \cdots, \tag{6.3}$$

whose component $\mathscr{A}_0$ is equal to the field $\mathbf{k}$ and whose components $\mathscr{A}_k$ for $k > 0$ consist of the formal homogeneous polynomials of degree k in vectors from M.

Consider the commutative algebra A with identity element e generated as an algebra by the elements of some $\mathbf{k}$-linear subspace $V \subset A$. Fix a $\mathbf{k}$-linear map $\pi\colon M \to V$ of M onto V. Then π extends to an algebra homomorphism $\pi\colon \mathscr{A} \to A$ of $\mathscr{A} = \mathscr{S}(M)$ onto A.

Theorem 6.1.11 has the following consequence.

Theorem 6.1.14. *If the algebra A possesses Gorenstein duality, then it is isomorphic to $\mathscr{S}(M)_{\mathscr{L}}$ for some $\mathbf{k}$-linear function $\mathscr{L}$ on the symmetric algebra $\mathscr{S}(M)$ of M.*

We can see that describing algebras with Gorenstein duality reduces to describing the dual space of the algebra $\mathscr{S}(M)$. In what follows we present two different descriptions of $\mathscr{S}(M)^*$, which give rise to two different descriptions of algebras with Gorenstein duality.

6.1.5. *Formal series of symmetric forms on M.* Let $T[M]$ be the space of formal series

$$B = B_0 + B_1 + \dots + B_k + \cdots,$$

where the kth term B_k is a $\mathbf{k}$-multilinear symmetric k-form on M. We show that $(\mathscr{S}(M))^*$ can be identified with $T[M]$.

As a $\mathbf{k}$-linear space, $\mathscr{S}(M)$ is spanned by the monomials $y_1 \cdots y_k$, where $y_1, \dots, y_k$ are all possible unordered k-tuples of elements of a basis of M. We define a pairing of $T[M]$ and $\mathscr{S}(M)$.

Definition 6.1.15. The pairing of a series $B \in T[M]$, where $B = B_0 + B_1 + \dots + B_k + \cdots$, and a monomial $y_1 \cdots y_k$ is the value $B_k(y_1, \dots, y_k)$ of the k-form B_k on the k-tuple $(y_1, \dots, y_k)$. It extends to the whole of $\mathscr{S}(M)$ by linearity.

The following result is straightforward.

Lemma 6.1.16. *The above pairing of $T[M]$ and $\mathscr{S}(M)$ defines an isomorphism between $T[M]$ and the dual space of $\mathscr{S}(M)$. Each linear function on $\mathscr{S}(M)$ that vanishes on all components $\mathscr{S}(M)_k$ of order $k > n$ is given by a pairing with a finite sum of symmetric forms $B = B_0 + \dots + B_n$.*

We describe the algebra $\mathscr{S}(M)_{\mathscr{L}(B)}$ constructed from a function $\mathscr{L}(B)$ which corresponds to a series $B = B_0 + \cdots + B_k + \cdots$ whose terms B_k are symmetric k-forms on M. By definition, the ideal $I_{\mathscr{L}(B)} \subset \mathscr{S}(M)$ constructed from $\mathscr{L}(B)$ consists of the elements $w \in \mathscr{S}(M)$ such that the operator (w) vanishes at B (that is, $w \in I_{\mathscr{L}(B)}$ if and only if all terms of the formal series $\rho(w)(B)$ vanish identically).

Theorem 6.1.17. *The algebra $\mathscr{S}(M)_{\mathscr{L}(B)}$ is the quotient algebra of $\mathscr{S}(M)$ with respect to the ideal $I_{\mathscr{L}(B)}$ of elements w such that the series $\rho(w)(B)$ is identically zero.*

The kth power $\mathbf{m}^k$ of the maximal ideal $\mathbf{m} \subset \mathscr{S}(M)_{\mathscr{L}(B)}$ is zero if and only if B is a finite sum, $B = B_0 + \cdots + B_{k-1}$.

For another description of $(\mathscr{S}(M))^*$ we need some properties of polynomials over an (infinite-dimensional) space M, which we discuss in the next subsection.

6.1.6. *Polynomials on infinite-dimensional spaces.* Here we recall the definition of the ring of polynomials $\mathbf{k}[M]$ over an (infinite-dimensional) $\mathbf{k}$-linear space M. This ring is an algebraic generalization of polynomial rings over finite-dimensional spaces. Unless specified otherwise, $\mathbf{k}$ will be the field of complex numbers $\mathbb{C}$, the field of real numbers $\mathbb{R}$, or the field of rationals $\mathbb{Q}$.

An infinite-dimensional space M is often endowed with some topology, and considerations are limited to continuous polynomials in this topology. The algebraic definition needed here corresponds to the weakest reasonable topology. A subset of M is open in this topology if its intersection with each finite-dimensional subspace is open (in the topology of the subspace).

In § 6.1.7 we recall that this weak topology on M is sufficient to apply the usual apparatus of differential calculus to polynomials $P \in \mathbf{k}[M]$. We shall use this apparatus to interpret $\mathscr{S}(M)$ as the algebra of differential operators on the polynomial ring.

Definition 6.1.18. A function $F\colon M \to \mathbf{k}$ is called

– a *polynomial of degree* $\leqslant k$ if its restriction to each finite-dimensional subspace is a polynomial of degree $\leqslant k$;

– a *homogeneous polynomial of degree k* if its restriction to each finite-dimensional subspace is a homogeneous polynomial of degree k (the function identically equal to zero is a homogeneous polynomial of degree k for each k);

– a *polynomial* if for some k it is a polynomial of degree at most k.

The polynomials on M form a ring $\mathbf{k}[M]$ with respect to the natural operations of addition and multiplication. The polynomials of degree $\leqslant k$ and homogeneous polynomials of degree k form $\mathbf{k}$-linear spaces.

The usual apparatus of differential calculus can be applied to polynomials $P \in \mathbf{k}[M]$.

Definition 6.1.19. The *Gâteaux derivative* $F'_v(x)$ of a function F at a point $x \in M$ in the direction of $v \in M$ is

$$F'_v(x) = \lim_{t\to 0} \frac{F(x+tv) - F(x)}{t}$$

whenever this limit exists.

We now formulate the properties of polynomials that will be needed.

Each polynomial has a Gâteaux derivative at each point x and in each direction v. (To prove this it suffices to use the differentiability properties of the restriction of the polynomial to the plane L spanned by x and v.)

If F is a polynomial of degree $\leqslant k$ (or a homogeneous polynomial of degree k), then, as a function of x, the derivative $F'_v(x)$ is a polynomial of degree $\leqslant k-1$ (a homogeneous polynomial of degree $k-1$, respectively).

For each point x the derivative $F'_v(x)$ depends on the direction v in a linear way.

The linear function D_xF defined by

$$D_xF(v) = F'_v(x),$$

is called the differential of F at x.

Let v_1 and v_2 be two vectors. At each point x the second derivative $F^{(2)}_{v_1,v_2}(x)$ is a symmetric bilinear form of v_1 and v_2. The *second differential D_x^2F of F at x* is the quadratic form corresponding to the bilinear form $F^{(2)}_{v_1,v_2}(x)$, that is,

$$D_x^2F(u) = F^{(2)}_{u,u}(x).$$

Before we proceed to higher-order differentials, we recall the definition of the polarization of a homogeneous polynomial.

Definition 6.1.20. A symmetric k-form B is called a *polarization* of a homogeneous polynomial P of degree k if $P(u) = B(u, \dots, u)$ for each vector u.

Let $v_1, \dots, v_m$ be an m-tuple of vectors. At each point x the mth derivative $F^{(m)}_{v_1,\dots,v_n}(x)$ is a symmetric multilinear function of $v_1, \dots, v_m$.

Definition 6.1.21. The differential D_x^mF at x is a homogeneous polynomial of degree m with polarization $F^{(m)}_{v_1,\dots,v_n}(x)$, that is, $D_x^mF(u) = F^{(m)}_{u,\dots,u}(x)$.

Theorem 6.1.22 (Taylor's formula for polynomials). *For a polynomial $F\colon M \to \mathbf{k}$ of degree $\leqslant k$ and a pair of points $x, u \in M$,*

$$F(x+u) = F(x) + D_xF(u) + \frac{1}{2}D_x^2F(u) + \dots + \frac{1}{k!}D_x^kF(u). \tag{6.4}$$

Proof. Formula (6.4) reduces to Taylor's classical formula for the restriction $F\big|_L$ of F to the plane L spanned by the vectors x and u. □

Corollary 6.1.23. *A polynomial $F\colon M \to \mathbf{k}$ of degree $\leqslant k$ has a unique expansion into a sum of homogeneous polynomials of degree ranging from 0 to k. This expansion defines the structure of a graded ring on $\mathbf{k}[M]$.*

Proof. The existence of the expansion follows from Taylor's formula for the polynomial F at $x = 0$. Uniqueness is a consequence of the uniqueness of the Taylor polynomial for the restrictions of F to one-dimensional subspaces. □

Corollary 6.1.24. *A homogeneous polynomial F of degree k has a unique polarization B. It is given by*

$$B(v_1, \dots, v_k) = \frac{1}{k!}F^{(k)}_{v_1,\dots,v_k}. \tag{6.5}$$

Proof. By Taylor's formula for the homogeneous polynomial F at 0, we have $F(u) = D_0^k F(u)/k!$. The form $F^{(k)}_{v_1,\dots,v_k}$ is a polarization of $D_0^k F$. Hence the k-form B defined by (6.5) is a polarization of F. The fact that the polarization is unique can be verified using induction on the degree of the homogeneous polynomial F. □

Formula (6.5) defines a canonical isomorphism between the space of symmetric k-forms and the space $\mathbf{k}[M]_k$ of homogeneous polynomials of degree k. Another version of formula (6.5) for the polarization of a homogeneous polynomial, which involves finite differences instead of differentiations, can sometimes also be of use.

Lemma 6.1.25. *For any point $x \in M$ the polarization B of a homogeneous polynomial F of degree k satisfies*

$$B(v_1, \dots, v_k) = \frac{1}{k!} \sum_{0 \leqslant j \leqslant k} \Bigl((-1)^{k-j} \sum_{1 \leqslant i_1 < \dots < i_j \leqslant k} F(x + v_{i_1} + \dots + v_{i_j}) \Bigr). \tag{6.6}$$

Example 6.1.26. For $k = 2$ formula (6.6) produces the following expression for the symmetric bilinear form B in terms of the associated quadratic form F:

$$B(v_1, v_2) = F(x + v_1 + v_2) - F(x + v_1) - F(x + vv_2) + F(x).$$

We sketch the proof of Lemma 6.1.25 for $\mathbf{k} = \mathbb{R}$. By Lagrange's theorem (for the restriction of F to the subspace spanned by vectors $x, v_1, \dots, v_k$), there exist numbers $0 < \varepsilon_1 < 1, \dots, 0 < \varepsilon_k < 1$ such that

$$F^{(k)}_{v_1,\dots,v_k}(x+\varepsilon_1 v_1+\dots+\varepsilon_k v_k) = \frac{1}{k!} \sum_{0 \leqslant j \leqslant k} \Bigl((-1)^{k-j} \sum_{1 \leqslant i_1 < \dots < i_j \leqslant k} F(x+v_{i_1}+\dots+v_{i_j}) \Bigr).$$

Each kth partial derivative of a polynomial F of degree k is a constant, hence

$$F^{(k)}_{v_1,\dots,v_k}(x + \varepsilon_1 v_1 + \dots + \varepsilon_k v_k) = F^{(k)}_{v_1,\dots,v_k}(0).$$

This yields the claim of Lemma 6.1.25 for $\mathbf{k} = \mathbb{R}$.

6.1.7. *Formal series of polynomials and the algebra* $\mathrm{Diff}(M)$. Let $\mathbf{k}\{\{M\}\}$ be the space of formal series

$$P = P_0 + P_1 + \dots + P_k + \cdots,$$

where the kth term P_k is a homogeneous polynomial of degree k over M. We define the commutative algebra $\mathrm{Diff}(M)$ of linear differential operators with constant coefficients that act on $\mathbf{k}\{\{M\}\}$.

Definition 6.1.27. The algebra $\mathrm{Diff}(M)$ is the operator algebra on the ring of series $\mathbf{k}\{\{M\}\}$ that is generated by the operators of multiplication by constants in $\mathbf{k}$ and the operators of differentiation along vectors $v \in M$.

Now we define a representation $\mathscr{D}\colon \mathscr{S}(M) \to \mathrm{Diff}(M)$ of $\mathscr{S}(M)$ in $\mathrm{Diff}(M)$.

Definition 6.1.28. 1) For $w = \lambda \in S^0(M) = \mathbf{k}$ the series $\mathscr{D}(\lambda)(P)$ is equal to λP.

2) For $w = v \in M = S^1(M)$ the series $\mathscr{D}(v)P$ is equal to the derivative P'_v of P along v.

The equalities in 1) and 2) produce a representation of $\mathscr{D}$ because the algebra $\mathscr{S}(M)$ is generated by the components $\mathscr{S}^0(M)$ and $\mathscr{S}^1(M)$.

It is easy to see that $\mathscr{D}$ establishes an algebra isomorphism between $\mathscr{S}(M)$ and $\mathrm{Diff}(M)$. We define a pairing of the spaces $\mathbf{k}\{\{M\}\}$ and $\mathscr{S}(M)$.

Definition 6.1.29. The pairing of a series $P \in \mathbf{k}\{\{M\}\}$ and an element $w \in \mathscr{S}(M)$ is defined as the free term of the series $\mathscr{D}(w)P$.

The following assertion is easy to verify.

Lemma 6.1.30. *The pairing of* $\mathbf{k}\{\{M\}\}$ *and* $\mathscr{S}(M)$ *determines an isomorphism between* $\mathbf{k}\{\{M\}\}$ *and the dual space of* $\mathscr{S}(M)$*. Each linear function on* $\mathscr{S}(M)$ *that vanishes on all the components* $\mathscr{S}(M)_k$ *of order* k *greater than* n *is given by a pairing of polynomials of degree at most* n*.*

Let $\Lambda\colon T[M] \to \mathbf{k}\{\{M\}\}$ denote the map taking a series $B = B_0 + B_1 + \cdots + B_k + \cdots$ to the series $P = P_0 + P_1 + \cdots + P_k + \cdots$ such that P_k is a homogeneous polynomial of order k given as the restriction of the k-form B_k to the diagonal, divided by $k!$, that is,

$$P_k(x) = \frac{1}{k!} B_k(x, \ldots, x). \tag{6.7}$$

Lemma 6.1.31. *The two series* $B \in T[M]$ *and* $\Lambda B \in \mathbf{k}\{\{M\}\}$ *produce the same linear function on* $\mathscr{S}(M)$ *when* $\mathscr{S}(M)^*$ *is identified with* $T[M]$ *and* $\mathbf{k}\{\{M\}\}$*.*

Proof. By Corollary 6.1.24, the value of the linear function corresponding to the k-form B_k at a homogeneous element $w = v_1 \cdots v_k \in \mathscr{S}^k(M)$ is equal to the value of the linear function corresponding to the polynomial $P_k = \Lambda B_k$ at the same element. This equality extends to all elements of $T[M]$ by linearity. $\square$

Now we describe the algebra $\mathscr{S}(M)_{\mathscr{L}(P)}$ constructed for the function $\mathscr{L}(P)$ corresponding to a series $P = P_0 + P_1 + \cdots + P_k + \cdots$. We start by describing the ideal $I_{\mathscr{L}(P)} \subset \mathscr{S}(M)$ constructed from $\mathscr{L}(P)$.

Lemma 6.1.32. *The ideal* $I_{\mathscr{L}(P)}$ *consists of those* $w \in \mathscr{S}(M)$ *for which the operator* $\mathscr{D}(w)$ *vanishes at* P *(that is,* $w \in I_{\mathscr{L}(P)}$ *if and only if all the terms in the formal series* $\rho(w)(P)$ *vanish identically).*

Theorem 6.1.33. *The algebra* $\mathscr{S}(M)_{\mathscr{L}(P)}$ *is the quotient algebra of* $\mathscr{S}(M)$ *with respect to the ideal* $I_{\mathscr{L}(P)}$ *of elements* w *such that the series* $\rho(w)(P)$ *vanishes identically.*

The k*th power* $\mathbf{m}^k$ *of the maximal ideal* $\mathbf{m} \in \mathscr{S}(M)_{\mathscr{L}(P)}$ *is zero if and only if* $P = P_0 + \cdots + P_k$ *is a polynomial.*

6.2. Homology of smooth toric varieties and rings of intersections.

6.2.1. *Projective toric varieties.* Projective toric varieties have a simple explicit description. The fan W of such a variety is dual to some n-dimensional lattice polytope Δ, that is, $W = \Delta^{\perp}$. The equality $W = \Delta^{\perp}$ does not define Δ uniquely. There are plenty of polytopes with the same dual fan.

With each polytope Δ such that $W = \Delta^{\perp}$ we can associate a map of the variety M_W to a projective space defined as follows. Suppose that Δ contains N lattice points, $N = \#(\Delta \cap \mathbb{Z}^n)$. Consider the projective space $\mathbb{CP}^{N-1}$ with coordinates

$x_1 : \cdots : x_N$. We number the characters $\chi_1, \ldots, \chi_N$ corresponding to lattice points in Δ in an arbitrary order. The *Kodaira map* $K_\Delta \colon (\mathbb{C}^*)^n \to \mathbb{CP}^{N-1}$ is defined by

$$K_\Delta(x) = \chi_1(x) : \cdots : \chi_N(x). \tag{6.8}$$

It has a regular extension to M_W and defines the required map of M_W into $\mathbb{CP}^{N-1}$. This map K_Δ is not an embedding, in general.

For sufficiently large polytopes the map $K_\Delta \colon M_W \to \mathbb{CP}^{N-1}$ *is an embedding.* We say that a polytope Δ is *sufficiently large* if condition A below holds for each vertex A of the polytope. Let $\Delta - A$ denote the parallel translation of Δ by the vector $-A$, and let $\Lambda(\Delta, A) \subset \mathbb{Z}^n$ be the semigroup of all lattice points in the minimal real cone containing $\Delta - A$.

Condition A. *The semigroup* $\Lambda(\Delta, A)$ *is generated by the points of intersection of* $\Delta - A$ *with the lattice* $\mathbb{Z}^n$.

Each lattice polytope Δ multiplied by a sufficiently large integer is a sufficiently large polytope (this integer can be taken equal to $n+1$).

For sufficiently large polytopes Δ *the map*

$$K_\Delta \colon M_W \to \mathbb{CP}^{N-1}$$

is an embedding. Moreover, the image of $K_\Delta(M_W)$ is equal to the closure of the image of $(\mathbb{C}^*)^n$ under the map (6.8). Using K_Δ, we define an action ρ of the group $(\mathbb{C}^*)^n$ on the projective space as

$$\rho(x) \colon (u_1 : \cdots : u_N) \mapsto (\chi_1(x)u_1 : \cdots : \chi_N(x)u_N).$$

The closure of the image $K_\Delta((\mathbb{C}^*)^n)$ of the torus under the Kodaira map, endowed with the torus action ρ, is a projective toric variety $M_{\Delta^\perp}$ isomorphic to the original variety $M_{\Delta^\perp}$.

The restriction of ρ to the real torus $T^n \subset (\mathbb{C}^*)^n$ preserves the standard Kähler metric in the projective space. This action is connected with the *moment map* $\Upsilon \colon \mathbb{CP}^{N-1} \to \mathbb{R}^n$, which is defined up to adding a constant vector $a \in \mathbb{R}^n$. (The moment map takes its values in the Lie coalgebra of the group T^n, which has a natural identification with the character space $\mathbb{R}^n$.)

The restriction of Υ to $M_{\Delta^\perp} \subset \mathbb{CP}^{N-1}$ is of special interest. For a suitable choice of the additive constant $a \in \mathbb{R}^n$ it takes $M_{\Delta^\perp}$ onto the polytope Δ. Then the orbit O_Γ corresponding to the dual cone $\sigma^\Gamma \in \Delta^\perp$ of a face $\Gamma \subset \Delta$ is taken to the interior Γ° of Γ (in the topology of the affine space spanned by Γ). In addition, Υ defines a locally trivial fibration of the orbit O_Γ over Γ°, with the real torus T^k, where $k = \dim_{\mathbb{C}} O_\Gamma$, being the fibre.

6.2.2. *Linear function on a simple polytope.* Recall that a polytope $\Delta \subset \mathbb{R}^n$ is said to be *simple* if each of its vertices is incident to precisely n facets. For each vertex A of such a polytope there exists an affine transformation taking A to the origin and a neighbourhood of A on Δ to a neighbourhood of the origin in the positive octant $\mathbb{R}^n_{\geqslant 0}$.

We say that a linear function $\xi \colon \mathbb{R}^n \to \mathbb{R}$ is *generic* if it is not constant on any edge of Δ. The *index* of a generic linear function ξ at a vertex $A \in \Delta$ is the number of edges going out of A such that ξ is decreasing along each edge. We say that a face $\Gamma \subset \Delta$ of dimension i is *ξ-distinguished* if it contains a vertex A with index i and the i edges issuing from A along which ξ is decreasing. The restriction of ξ to Γ takes its maximum value at A.

The following objects are connected with the polytope Δ:

– the *f-vector* $(f_0, f_1, \dots, f_n)$ whose component $f_i = f_i(\Delta)$ is the number of i-faces of Δ for each $i = 0, 1, \dots, n$ (we set f_n to be equal to 1);

– the *F-polynomial* $F(t) = f_0 + f_1 t + \cdots + f_n t^n$;

– the *H-polynomial* defined by $H(t) = F(t-1)$;

– the *h-vector* $(h_0, h_1, \dots, h_n)$ whose components $h_i = h_i(\Delta)$, $i = 0, 1, \dots, n$, are the coefficients of the H-polynomial, that is, $H(t) = h_0 + h_1 t + \cdots + h_n t^n$.

Lemma 6.2.1. *For each generic linear function ξ on a simple polytope Δ the number $h_i(\xi)$ of vertices with index i is equal to $h_i(\Delta)$. The number of ξ-distinguished i-faces in Δ is also equal to $h_i(\Delta)$.*

Proof. Consider the map φ from the set of faces of Δ to its vertex set that assigns to a face Γ the vertex $A \in \Gamma$ at which the restriction of ξ to Γ takes its maximum value. If $A \in \Delta$ has index i with respect to ξ, then $\varphi^{-1}(A)$ contains $\binom{i}{k}$ k-faces for each $k = 0, 1, \dots, n$. Hence

$$f_k = \sum_i \binom{i}{k} h_i(\xi).$$

Taken together, these equalities mean that $h_k(\xi) = h_k(\Delta)$. The map φ defines a one-to-one correspondence between the ξ-distinguished i-dimensional faces and the vertices with index i. □

Corollary 6.2.2 (Dehn–Sommerville theorem). *For a simple n-dimensional polytope Δ, its h-vector has the following properties*:

1) $h_i(\Delta) = h_{n-i}(\Delta)$ *for* $0 \leqslant i \leqslant n$;

2) $h_i(\Delta) \geqslant 1$ *for* $0 \leqslant i \leqslant n$;

3) $h_0(\Delta) = h_n(\Delta) = 1$. (*This is equivalent to the fact that the Euler characteristic* $\sum_{0 \leqslant i \leqslant n} (-1)^i f_i(\Delta)$ *of the simple polytope Δ is equal to* 1.)

Proof. 1) If a vertex $A \in \Delta$ has index i with respect to a generic linear function ξ, then it has index $n-i$ with respect to $-\xi$. Hence $h_i(\xi) = h_{n-i}(-\xi)$. Now property 1) follows from Lemma 6.2.1.

2) For each $0 \leqslant i \leqslant n$ and each vertex A we can find a generic linear function ξ such that A has index i with respect to ξ. Therefore, $h_i(\xi) \geqslant 1$.

3) Each generic linear function ξ attains its maximum value at a unique vertex of Δ, so $h_n(\xi) = 1$. The function ξ attains its minimum at a unique vertex of Δ, so $h_0(\xi) = 1$. By Lemma 6.2.1, this means that $h_0(\Delta) = h_n(\Delta) = 1$. By definition, the value of H at 0 is $h_0(\Delta)$. Since $F(-1) = H(0)$, it follows that

$$\sum_{0 \leqslant i \leqslant n} (-1)^i f_i(\Delta) = h_0(\Delta) = 1. \ \square$$

In the next subsection we explain that, for any given integrally simple polytope Δ, there is an associated smooth projective variety with odd Betti numbers equal to zero and even Betti numbers b_{2i} equal to h_i. For these polytopes the equality $h_i = h_{n-i}$ follows from Poincaré duality for the corresponding toric variety. On a smooth manifold Poincaré duality can be established by means of its representations as two CW-complexes, one of which is constructed by using an arbitrary Morse function f and the other one by using $-f$. The definition of the index of a vertex of a simple polytope with respect to a generic linear function and the proof of the Dehn–Sommerville theorem presented above have been motivated by Morse theory and the proof of Poincaré duality by means of this theory. Developing this idea made it possible to obtain new results on the combinatorics of polytopes, which have led to a solution of the old problem concerning groups generated by reflections in multidimensional Lobachevskii spaces [37].

6.2.3. *The homotopy type of a smooth projective toric variety.* A toric variety M_W is smooth and projective if and only if its fan W is convex and non-degenerate, that is, it has the form $\Delta^\perp$, where Δ is an integrally simple polytope (see Definition 2.3.10). In what follows we consider varieties M_W of this kind. Fix an integrally simple polytope Δ such that $W = \Delta^\perp$. Below, in discussing M_W we mention the polytope Δ and its faces repeatedly. The map K_Δ embeds M_W into a projective space. The composition $\xi \circ \Upsilon$ of the moment map and a generic linear function is a Morse function on $M_{\Delta^\perp} = K_\Delta(M_W)$. Its critical points are zero-dimensional orbits on $M_{\Delta^\perp}$. Furthermore, the Morse index of $\xi \circ \Upsilon$ at a zero-dimensional orbit $B \in M_{\Delta^\perp}$ is $2i$, where i is the index of the vertex $A = \Upsilon(B) \in \Delta$ for the linear function ξ.

Theorem 6.2.3. *The variety M_W, where $W = \Delta^\perp$ and Δ is an integrally simple polytope, has the homotopy type of a CW-complex with only even-dimensional cells such that the number of $2i$-dimensional cells is $h_i(\Delta)$. The groups $H_{2i}(M_W, \mathbb{Z})$ are torsion free and the Betti numbers $b_{2i}(M_W)$ are equal to $h_i(\Delta)$.*

Proof. It is sufficient to consider $\xi \circ \Upsilon$ as a Morse function on M_W and use the above geometric description of its critical points and their indices. □

Corresponding to each i-dimensional face Γ_k of Δ there is an i-dimensional complex cycle $T(\Gamma_k)$ in M_W such that $\dim_{\mathbb{R}} T(\Gamma_k) = 2i$. Linear combinations of the $T(\Gamma_k)$ generate the whole group $H_{2i}(M_W, \mathbb{Z})$. Morse theory also provides natural bases in the homology groups.

Theorem 6.2.4. *For $W = \Delta^\perp$ the group $H_{2i}(M_W, \mathbb{Z})$ is generated by the i-dimensional toric subvarieties $T(\Gamma_k)$ of M_W that are the closures of the orbits O_{Γ_k} of dimension $\dim_{\mathbb{C}} O_{\Gamma_k} = i$ corresponding to ξ-distinguished faces Γ_k of Δ with $\dim_{\mathbb{R}} \Gamma_k = i$.*

Theorem 6.2.4 has a stronger version, which also has a simpler proof.

Definition 6.2.5. A smooth compact complex algebraic variety has the structure of an *algebraic CW-complex* if it can be represented as a union of disjoint cells of the following form. Each cell is an algebraic variety isomorphic to an affine space $\mathbb{C}^i$, whose dimension depends on the cell. The closure of each cell is a union of this cell with some lower-dimensional ones.

All smooth compact manifolds have the homotopy type of a finite CW-complex. However, it is only in some exceptional cases that a compact algebraic variety has the structure of an algebraic CW-complex. For example, the projective line $\mathbb{CP}^1$ is the only connected algebraic curve with this structure.

The Chow ring of an algebraic CW-complex is isomorphic (upon a change of grading) to its cohomology ring. For general algebraic varieties this is far from true. For example, the zeroth Chow group of a connected algebraic curve of genus g contains the Jacobian variety of this curve, which is a complex Abelian g-dimensional torus.

Theorem 6.2.6. *Let ξ be a generic linear function on an integrally simple polytope Δ. Then the variety M_W, where $W = \Delta^\perp$, has the structure of an algebraic CW-complex whose cells are in a bijective correspondence with the vertices of Δ. Furthermore, the $2i$-dimensional cells correspond to the vertices A with index i with respect to ξ. These cells consist of the orbits corresponding to the faces Γ of Δ such that the restriction of ξ to Γ attains its maximum value at A.*

Proof. Let $\tau \in (\mathbb{R}^n)^*$ be the vector in the Lie algebra of the group $(\mathbb{C}^*)^n$ that determines $-\xi$, so that $-\xi(x) = \langle \tau, x \rangle$. We can assume without loss of generality that τ is an integer vector which corresponds to an algebraic one-parameter group. The action of this group on the toric variety is similar to the action of the gradient flow of $-\xi$, but it is much easier to describe (one does not need the Kähler metric or the moment map). Let us show that the variety is partitioned by the action of the one-parameter group into cells whose points tend to zero-dimensional orbits, which are different for different cells. Furthermore, the cell whose points tend to the zero-dimensional orbit corresponding to a vertex A is the union of the orbits described in the theorem.

Indeed, let $A_j \in \Delta$ be the lattice point closest to A on the jth edge incident to A. Consider the affine toric subvariety containing the zero-dimensional orbit corresponding to $A \in \Delta$. This subvariety is equivariantly isomorphic to the standard space $\mathbb{C}^n$ endowed with the induced action of the torus $(\mathbb{C}^*)^n$. The character corresponding to the lattice point $A_j - A$ is the jth coordinate function on $\mathbb{C}^n$. The one-parameter group defines a linear diagonal action on $\mathbb{C}^n$. It multiplies the jth coordinate by t^{k_j}, where t is the group parameter and $k_j = \langle \tau, A_j - A \rangle$.

Hence k_j is positive if and only if $-\xi$ is increasing along the jth edge of the polytope as one moves away from A, and therefore ξ is decreasing. The points tending to the origin under the action of the one-parameter group are precisely those that lie in the coordinate subspace such that for all the coordinates in this subspace the corresponding numbers h_j are positive. □

Lemma 6.2.1 and the Dehn–Sommerville theorem extend immediately to simplicial fans not necessarily dual to simple convex polytopes.

In exactly the same way, using the action of a sufficiently generic one-parameter group, we can show that the Grassmannians $G(n, k)$ of k-dimensional subspaces of $\mathbb{C}^n$ and some other varieties of this type have the structure of an algebraic CW-complex. In the same way we can show that smooth compact non-projective toric varieties have the structure of an algebraic CW-complex.

6.2.4. *Cycles corresponding polytope faces.* Instead of selecting basis cycles among the cycles $T(\Gamma_k)$ such that $\dim_{\mathbb{R}} \Gamma_k = i$, we can describe all relations between the $T(\Gamma_k)$. These are consequences of the construction described in what follows, which will also be needed later on.

We say that a lattice *polytope* Δ *is subordinate to a fan* W if its support function H_Δ is linear on each cone $|\sigma|$ of W. Suppose that the Newton polytope $\Delta(P)$ of a Laurent series P is subordinate to the fan W of a smooth projective variety M_W. Consider the algebraic variety $X \subset (\mathbb{C}^*)^n$ given by the equation $P = 0$. We can regard the closure $\overline{X}$ of X in M_W as a cycle in the group $H_{2n-2}(M_W, \mathbb{Z})$. For different Laurent polynomials P with the same Newton polytope these cycles are homologous. Moreover, the following result holds.

Theorem 6.2.7. *The cycle* $\overline{X}$ *is homologous to the cycle* $\sum k_i(\Delta(P))T(\Gamma_i)$, *where the* Γ_i *are the facets and* $k_i = -H_{\Delta(P)}(\xi_i)$. *(Here* ξ_i *is the irreducible integer vector on the dual ray* $\sigma_{\Gamma_i} \subset W$ *of* Γ_i.*)*

Proof. We can regard the Laurent polynomial P as a rational function on M_W. On each divisor $T(\Gamma_i)$ the function P has order $k_i(\Delta) = -H_{\Delta(P)}(\xi_i)$. By definition, the principal divisor of P on the torus $(\mathbb{C}^*)^n$ is X, so the principal divisor of P on M_W is $\overline{X} - \sum k_i(\Delta)T(\Gamma_i)$, which yields the theorem. □

Corollary 6.2.8. *For each character* χ *of* $(\mathbb{C}^*)^n$,

$$\sum \langle \xi_i, \chi \rangle T(\Gamma_i) = 0. \tag{6.9}$$

Proof. This follows from Theorem 6.2.7 for $P = \chi$. □

Each face Γ of Δ is, in turn, an integrally simple polytope $\widetilde{\Delta}$ of dimension $k = \dim_{\mathbb{R}} \Gamma$. The polytope $\widetilde{\Delta} = \Gamma$ is associated with the smooth toric variety $M(\widetilde{\Delta})$ of complex dimension k. Each $(k-1)$-face of $\widetilde{\Gamma}_i$ corresponds to an invariant cycle $\widetilde{T}(\widetilde{\Gamma}_i) \subset M(\widetilde{\Delta})$ in $\widetilde{\Delta}$. The cycles $\widetilde{T}(\widetilde{\Gamma}_i)$ in $M(\widetilde{\Delta})$ are connected by relations similar to the relations (6.9) for the cycles $T(\Gamma_i)$ in M_W. The faces $\widetilde{\Gamma}_i$ of $\widetilde{\Delta} = \Gamma$ are also faces of the original polytope Δ. The corresponding cycles $\widetilde{T}(\widetilde{\Gamma}_i)$ in $M_{\mathbb{Z}}$ and $T(\widetilde{\Gamma}_i)$ in M_W are isomorphic and connected by the same relations.

Now we consider this in more detail.

A face $\Gamma \subset \Delta$ with $\dim \Gamma = k$ corresponds to a linear k-dimensional space $L_\Gamma \subset \mathbb{R}^n$ parallel to the affine subspace spanned by Γ. The subspace L_Γ contains the integer lattice $L_\Gamma \cap \mathbb{Z}^n$ of rank k. We can view L_Γ as the character space of the quotient torus $(\mathbb{C}^*)^{\dim \Gamma}$ of the original torus $(\mathbb{C}^*)^n$ with respect to the connected subtorus whose Lie algebra is the complexification of the minimal subspace of $(\mathbb{R}^n)^*$ containing the cone σ^Γ.

We can move $\Gamma = \widetilde{\Delta}$ to L_Γ by parallel translation. The polytope $\widetilde{\Delta} = \Gamma \subset L_\Gamma$ corresponds to a smooth projective toric variety $M_{\mathbb{Z}}$ with fan Z equal to $\Gamma^\perp \subset L_\Gamma^*$. Each $(k-1)$-dimensional face of $\widetilde{\Gamma}_i$ in Γ corresponds to an irreducible integer covector $\widetilde{\xi}_i$ in the ray of Z dual to $\widetilde{\Gamma}_i$. The space L_Γ^* is the quotient of $(\mathbb{R}^n)^*$ with respect to the minimal linear subspace $L_\Gamma^\perp \subset (\mathbb{R}^n)^*$ containing the cone $\sigma(\Gamma)$. Let π denote the map from $(\mathbb{R}^n)^*$ to L_Γ^*. Let $\xi \in (\mathbb{Z}^n)^*$ be an arbitrary covector such that $\pi(\xi) = \widetilde{\xi}$.

Corollary 6.2.9. *For each character χ of the torus $(\mathbb{C}^*)^{\dim\Gamma}$, the following relation holds between the invariant $(k-1)$-dimensional cycles $T(\widetilde{\Gamma}_i)$ corresponding to the $(k-1)$-dimensional faces $\widetilde{\Gamma}_i \subset \Gamma$:*

$$\sum \langle \xi_i, \chi\rangle T(\widetilde{\Gamma}_i) = 0.$$

Each vertex $A \in \Gamma$ corresponds to a zero-dimensional orbit $O_A \in M_W$ and k invariant $(k-1)$-dimensional cycles $T(\widetilde{\Gamma}_i) \subset T(\Gamma)$ containing O_A.

Corollary 6.2.10. *Invariant $(k-1)$-dimensional cycles containing O_A are integer linear combinations of invariant $(k-1)$-dimensional cycles in $M_{\Gamma^\perp}$ not containing O_A.*

Proof. To simplify the notation we carry out the proof for $\Gamma = \Delta$ and $k = n$. Since Δ is a simple polytope, there exist vectors $\xi_1, \dots, \xi_n \in (\mathbb{Z}^n)^*$ generating the group $(\mathbb{Z}^n)^*$ and spanning the one-dimensional cones in W dual to the facets of Δ containing A. Now it is sufficient to use equality (6.9) for the characters $\chi_1, \dots, \chi_n$ such that $\langle \xi_i, \chi_j\rangle = \delta_i^j$. □

Theorem 6.2.11. *The group $H_{2i}(M_{\Delta^\perp}, \mathbb{Z})$ is isomorphic to the quotient group of linear combinations of i-dimensional toric subvarieties $T(\Gamma_i)$ of $M_{\Delta^\perp}$ that are the closures of i-dimensional orbits O_{Γ_i} with respect to the subgroup of relations in Corollary 6.2.9.*

Proof. Fix a linear function $\xi\colon \mathbb{R}^n \to \mathbb{R}$ which is generic relative to Δ. By the height of a face Γ we mean the maximum of the restriction of ξ to Γ. Assume that an i-dimensional face Γ_k is not ξ-distinguished. Then the cycle $T(\Gamma_k)$ is a linear combination of the cycles corresponding to faces with height less than that of Γ. Indeed, suppose that the restriction of ξ to Γ attains it maximum at a vertex A. Since the face Γ is not ξ-distinguished, there exists an edge l going out of A and not lying in Γ such that ξ is decreasing along l. Consider the $(k+1)$-dimensional face Ψ of Δ that contains Γ and l. We apply Corollary 6.2.10 to Ψ. By this corollary, $T(\Gamma_i)$ is a linear combination of cycles corresponding to i-dimensional faces of Ψ and not containing A. All these faces have height less than that of Γ. □

6.2.5. *The ring of intersections and faces of Δ.* Let M be a smooth compact real oriented n-dimensional manifold. Poncaré duality establishes an isomorphism between the homology and cohomology groups of M, namely $H_k(M, \mathbb{Z})$ and $H^{n-k}(M, \mathbb{Z})$ for $0 \leqslant k \leqslant n$.

Modulo torsion, the cup product in the cohomology ring has a clear geometric meaning when transferred to homology using Poincaré duality. Let us consider this in more detail, restricting generality to what is needed.

The manifolds M that we need (smooth compact toric varieties) are torsion-free, and the ring of intersections for such manifolds is defined in $H_*(M, \mathbb{Z})$. Furthermore, in these manifolds any class in each homology group $H_m(M, \mathbb{Z})$ has a representation of the form $\Gamma = \sum k_i \gamma_i$, where the γ_i are oriented m-dimensional manifolds with singularities of codimension 2. More precisely, each γ_i is a compact stratified set with a finite number of smooth strata of differing dimensions adjoining one another well enough. Moreover, the strata of highest dimension m are oriented, while the other strata have codimension at least 2.

With the exception of this subsection, throughout the paper we consider only cycles of the form $\Gamma = \sum k_i\gamma_i$, where the γ_i are algebraic subvarieties. All homology classes in toric varieties have such representations. For this reason, the ring of intersections of a toric variety can be described in purely algebraic terms (and the corresponding object is called the Chow ring of the toric variety; see § 4.1.5) and is meaningful not only for complex toric varieties, but also for varieties over an arbitrary algebraically closed field. On the other hand, for complex toric varieties the intersections of real cycles (and not only of complex algebraic ones) are governed by the ring of intersections of the variety.

To intersect two cycles Γ_1 and Γ_2 we must first replace them by homologous cycles $\Gamma_1' = \sum k_i\gamma_i$ and $\Gamma_2' = \sum m_j\delta_j$ such that for all i and j the strata of the components γ_i and δ_j are transversal to one another.

Then, we can define the intersection of the homology classes of Γ_1 and Γ_2 to be the homology class of the cycle $\sum k_i m_j \gamma_i \cap \delta_j$, where $\gamma_i \cap \delta_j$ is treated as a manifold with singularities endowed with the induced orientation.

We can show that the intersection of homology classes is well defined. (It is independent of the choice of transversal cycles homological to the original ones.) Moreover, the intersection in homology is Poincaré dual to the cup product in the cohomology ring.

Here are some facts on intersection theory in the homology on n-dimensional oriented manifolds with singularities. (We do not use these facts in what follows.) Assume that the manifold with singularities is compact and has a finite stratification with smooth strata of different dimensions adjoining one another well enough. In addition, the strata of highest dimension n are oriented and the other strata have codimension at least 2.

For such singular manifolds Goresky and MacPherson have introduced an entire spectrum of homology groups of various types. The cycles in these homology groups lie mainly in non-singular n-dimensional strata of the manifold. The intersections of a cycle with other strata have positive codimension in this cycle. The types of these homology groups differ depending on how large these codimensions can actually be. (The codimensions of intersections of chains are subject to the same restrictions as the codimensions of intersections of cycles.) In certain precisely described cases the intersection of cycles from homology groups of two (generally speaking, different) types is well defined and belongs to a homology group of some third type.

One homology type, which we call *exceptional*, admits cycles such that their intersection with each singular stratum is a set whose codimension in the cycle is no less than the codimension of the stratum. The intersection of two cycles of exceptional type is a well-defined operation, whose result remains in the class of cycles of exceptional type. Thus, a homology group of exceptional type is a ring with respect to the operation of intersection. The cohomology ring of a manifold with singularities is Poincaré dual to this ring of intersections.

Another homology type, which we call *standard*, admits cycles such that their intersection with each singular stratum is a set whose codimension in the cycle is at least two. The intersection of a cycle of standard type with a cycle of exceptional type is well defined and is a cycle of standard type. A homology group of standard type is isomorphic to the homology group of the singular manifold. The operation of intersection of cycles of standard type and cycles of exceptional type is Poincaré dual to the cap product of homology and cohomology classes of the singular manifold.

Although the definitions of various types of homology groups involve a stratification of the singular manifold, the homology groups do not depend on the choice of this stratification and are invariants of the manifold. Thus, the operation of intersecting cycles provides a geometric interpretation of the cup product in the cohomology ring and the cap product in homology and cohomology groups.

We return to toric varieties.

For a smooth toric variety M_W whose fan is dual to an integrally simple polytope Δ, that is, $W = \Delta^{\perp}$, the ring of intersections can be fully described in terms of Δ. Here we give an outline of such a description.

To each i-dimensional face Γ_k of Δ there corresponds a smooth algebraic subvariety $T(\Gamma_k) \subset M_W$ of complex dimension i. Integer linear combinations of the cycles $T(\Gamma_i)$ generate the group $H_{2i}(M_W, \mathbb{Z})$. This correspondence between the faces of the polytope and the cycles preserves the transversality property of intersections. (Here we say that *two faces* $\Gamma_i, \Gamma_j \subset \Delta$ *intersect transversally* if they either have an empty intersection or the face $\Gamma = \Gamma_i \cap \Gamma_j$ has dimension $\dim \Delta - \dim \Gamma_i - \dim \Gamma_j$.)

Theorem 6.2.12. *If* Γ_i *and* Γ_j *are disjoint faces, then the cycles* $T(\Gamma_i)$ *and* $T(\Gamma_j)$ *are also disjoint. On the other hand, if the faces intersect transversally and* $\Gamma = \Gamma_i \cap \Gamma_j$, *then the cycles* $T(\Gamma_i)$ *and* $T(\Gamma_j)$ *also intersect transversally and their intersection is* $T(\Gamma)$.

Proof. The cycles $T(\Gamma_i)$ and $T(\Gamma_j)$ are the unions of orbits corresponding to all the faces lying in Γ_i and Γ_j, respectively. Hence if $\Gamma_i \cap \Gamma_j = \varnothing$, then $T(\Gamma_i) \cap T(\Gamma_j) = \varnothing$.

Suppose that Γ_i and Γ_j intersect transversally and $\Gamma_i \cap \Gamma_j = \Gamma \neq \varnothing$. Then in the set of orbits lying in the intersection of $T(\Gamma_i)$ and $T(\Gamma_j)$ there is a unique orbit of maximum dimension. It corresponds to an open orbit in the affine toric variety M_σ, where σ is the closure of the open cone dual to the face Γ. The intersections of M_σ with $T(\Gamma_i)$, $T(\Gamma_j)$, and $T(\Gamma)$ are Zariski open subsets of these cycles. The variety M_σ is isomorphic to the affine space $\mathbb{C}^n$ minus all the coordinate subspaces not containing the open orbit in M_σ. The intersections of M_σ with $T(\Gamma_i)$ and $T(\Gamma_j)$ correspond to coordinate subspaces intersecting transversally in this orbit. □

Theorem 6.2.13. *If faces* $\{\Gamma_i\}$ (*of differing dimensions, in general*) *intersect transversally, then the corresponding cycles* $\{T(\Gamma_i)\}$ *also intersect transversally. In addition, the face* $\bigcap \Gamma_i$ *corresponds to the cycle* $\bigcap \{T(\Gamma_i)\}$.

We saw in the previous subsection that there are many relations between cycles of the form $T(\Gamma_i)$. It is easy to verify that these relations suffice to make the cycles transversal. In combination with Theorem 6.2.13, such constructions provide an explicit description of the ring of intersections.

Suppose that faces $\Gamma_1, \Gamma_2 \subset \Delta$ of dimensions m and k intersect non-transversally, the intersection being a face Γ of dimension l, so that $k + m - l < n$. Now we show how to replace the k-dimensional face Γ_2 by a linear combination of k-dimensional faces T_i such that, for each T_i, either $\dim T_i \cap \Gamma_2 = \varnothing$ or $\dim T_i \cap \Gamma_1 < l$. Then we must also replace the faces T_i that intersect Γ_1 non-transversally by linear combinations of faces whose intersections with Γ_1 have a still lower dimension, and so on.

Let ξ be a generic linear function on Δ. Let $A \in \Delta$ be the vertex of Γ at which ξ takes it maximum value on this face. In the set of edges outgoing from A, there are m edges in Γ_1, k edges in Γ_2, and l edges in Γ (the latter also lie in both Γ_1 and Γ_2). Since $n > m + k - l$, there exists an edge E incident to A that lies neither in Γ_1 nor in Γ_2. Let Γ_3 be the unique $(k + 1)$-dimensional face in Δ that contains Γ_2 and E. By construction, $\Gamma_2 \subset \Gamma_3$ and $\Gamma_1 \cap \Gamma_3 = \Gamma$. By Corollary 6.2.10, the cycle corresponding to Γ_2 can be represented as a linear combination of cycles corresponding to all the k-dimensional faces T_i of the polytope Γ_3 of dimension $k + 1$, with the exception of faces containing A. For each such T_i the intersection $T_i \cap \Gamma_1$ is either empty or a proper face of Γ, and therefore $\dim T_i \cap \Gamma_1 < l$.

6.2.6. *The ring of intersections of M_W and W-cycles in $(\mathbb{C}^*)^n$.* In this subsection we describe the ring of intersections of M_W using algebraic cycles in $(\mathbb{C}^*)^n$ and information about their closures in M_W. We assume that the fan W in $(\mathbb{R}^n)^*$ is dual to an integrally simple polytope $\Delta \subset \mathbb{R}^n$.

Definition 6.2.14. An algebraic variety $X \subset (\mathbb{C}^*)^n$ all of whose irreducible components have dimension k is called a *W-variety* if its closure $\overline{X}$ in M_W is disjoint from the orbits $O \subset M_W$ such that $\dim_{\mathbb{C}} O < n - k$.

Lemma 6.2.15. *The closure of a k-dimensional W-variety X and an $(n - k)$-dimensional orbit O are either disjoint or intersect in a finite set.*

Proof. If the set $O \cap \overline{X}$ is infinite, then it is non-compact because the affine variety O cannot contain a compact algebraic variety of positive dimension. Hence $\overline{X}$ intersects some of the orbits of dimension $< n - k$ adjacent to O, which is impossible. □

The following result has a similar proof.

Lemma 6.2.16. *The closure of a k-dimensional W-variety X intersects every orbit O in $M_{\Delta^\perp}$ in varieties of 'regular' dimension, that is, either $\overline{X} \cap O = \varnothing$ or $\dim \overline{X} \cap O = (k+m)-n$, where $k = \dim X$ and $m = \dim O$.*

A *stratification* of an algebraic variety X is a representation of X as the union $X = \bigcup X_i$ of disjoint smooth algebraic varieties X_i, which are called *strata*. We say that two varieties X and Y *intersect transversally* if they have stratifications $X = \bigcup X_i$ and $Y = \bigcup Y_j$ in which any two strata intersect transversally.

Let X and Y be transversally intersecting subvarieties of an n-dimensional variety M.

Lemma 6.2.17. *Assume that each irreducible component of X and Y has dimension k or m, respectively. If $X \cap Y \neq \varnothing$, then $k + m \geqslant n$ and each irreducible component of $X \cap Y$ has dimension $k + m - n$.*

Proof. The codimension of the intersection of two smooth subvarieties is the sum of their codimensions. We must show that $X \cap Y$ is the closure of the variety $X_k \cap Y_m$, where X_k is the union of all k-dimensional strata in X and Y_m is the union of all m-dimensional strata in Y. Indeed, at each point $a \in X \cap Y$ the germ of $X \cap Y$ has dimension $\geqslant k + m - n$. Hence each neighbourhood of a intersects $X_k \cap Y_m$. (The other strata in X and Y have intersections of dimension $\leqslant k + m - n$.) □

Theorem 6.2.18. *For any W-varieties X and Y there exists a Zariski open set $U \subset (\mathbb{C}^*)^n$ such that for all $g \in U$ the varieties X and gY intersect transversally within the torus $(\mathbb{C}^*)^n$, and for each orbit $O \subset M_W$ the varieties $O \cap \overline{X}$ and $O \cap g\overline{Y}$ intersect transversally within O.*

Proof. Consider the following stratifications of $\overline{X}$ and $\overline{Y}$. First, we represent $\overline{X}$ and $\overline{Y}$ as $\overline{X} = \bigcup (O \cap \overline{X})$ and $\overline{Y} = \bigcup (O \cap \overline{Y})$. For almost all $g \in (\mathbb{C}^*)^n$ the cycles X and gY intersect transversally within $(\mathbb{C}^*)^n$. By Lemma 6.2.16, the intersections $\overline{X} \cap O$ and $\overline{Y} \cap O$ of the cycles $\overline{X}$ and $\overline{Y}$ in M_W with any orbit O are subvarieties of 'regular' dimension. For almost all $g \in (\mathbb{C}^*)^n$ the cycles $\overline{X} \cap O$ and $g(\overline{Y} \cap O)$ intersect transversally within O (this is a general result in the theory of rings of conditions, which we have used above) and their intersection has dimension lower than $n - (k + m)$. Hence each irreducible component $Z \subset \overline{X} \cap g\overline{Y}$ of the intersection has dimension $n-(k+m)$, and there exists a Zarisky open set $U \subset (\mathbb{C}^*)^n$ intersecting Z such that $\overline{X} \cap U$ and $\overline{Y} \cap U$ are smooth varieties which intersect transversally, the intersection $Z \cap U$ being a smooth variety. Hence the cycles $\overline{X} \cap g\overline{Y}$ and $\overline{X} \cap \overline{Y}$ have the same images in the ring of intersections of M_W. □

Corollary 6.2.19. *For X, Y, and g as in Theorem 6.2.18,*

$$\overline{X \cap gY} = \overline{X} \cap \overline{gY}.$$

An integer linear combination of k-dimensional W-varieties is called a *k-dimensional W-cycle*. The following result is easy to verify.

Lemma 6.2.20. *Lemma 6.2.17 and Theorem 6.2.18 hold not only for W-varieties but also for W-cycles.*

6.2.7. *The ring of intersections and Newton polytopes.* Let P be a Laurent polynomial with Newton polytope Δ_1 that is subordinate to Δ (this means that the support function H_{Δ_1} is linear on each cone in the fan $\Delta^\perp$), and let $X \subset (\mathbb{C}^*)^n$ be given by the equation $P = 0$. The closure $\overline{X}(\Delta_1)$ of X in $M_{\Delta^\perp}$ can be regarded as a cycle in $H_{2n-2}(\Delta^\perp, \mathbb{Z})$. By Theorem 6.2.7, we have $\overline{X} = \sum k_i(\Delta) T(\Gamma_i)$, where the Γ_i are $(n-1)$-dimensional faces of Δ, $k_i = H_\Delta(\xi_i)$, and ξ_i is an irreducible integer vector in the dual ray $\sigma_{\Gamma_i} \subset \Delta^\perp$ of Γ_i.

With each system of lattice polytopes $\Delta_1, \ldots, \Delta_{n-k}$ subordinate to Δ we associate a variety X defined by sufficiently general systems of equations $P_1 = \cdots = P_{n-k} = 0$ with Newton polytopes $\Delta_1, \ldots, \Delta_{n-k}$. We call X a *non-degenerate complete intersection subordinate to* Δ.

Corollary 6.2.21. *The closure $\overline{X}$ in $M_{\Delta^\perp}$ of a non-degenerate complete intersection X subordinate to Δ is a cycle in $H_{2k}(M_{\Delta^\perp}, \mathbb{Z})$ which is equal to the intersection of the cycles $\overline{X}(\Delta_1), \ldots, \overline{X}(\Delta_{n-k})$.*

Theorem 6.2.22. *The intersection number of n cycles $\overline{X}(\Delta_1), \ldots, \overline{X}(\Delta_n)$ is the mixed volume of the polytopes $\Delta_1, \ldots, \Delta_n$ times $n!$.*

Proof. This is the toric version of the Kushnirenko–Bernstein theorem. □

Theorem 6.2.23. *Each cycle $\gamma \in H_{2n-2}(M_{\Delta^\perp}, \mathbb{Z})$ has a representation*

$$\gamma = \overline{X}_1(\Delta_1) - N\overline{X}_2(\Delta),$$

where N is a sufficiently large integer.

Proof. Let $\gamma = m_i T(\Gamma_i)$. We define a function H_γ on $(\mathbb{R}^n)^*$ as follows: H_γ is linear on each (simplicial) cone in the fan $\Delta^\perp$ and $H_\gamma(\xi_i) = m_i$ for irreducible integer vectors ξ_i lying on the rays in the one-dimensional skeleton of $\Delta^\perp$ that are dual to Γ_i. The function $H_\gamma + NH_\Delta$ is convex for a sufficiently large integer N, so it is the support function of a convex lattice polytope Δ_1 subordinate to Δ. We have $\gamma + N\overline{X}_2(\Delta) = \overline{X}_1(\Delta_1)$, which proves the theorem. □

Theorem 6.2.24. *Each cycle in $H_{2n-2k}(M_{\Delta^\perp}, \mathbb{Z})$ can be represented as an integer linear combination of the closures of non-degenerate $(n-k)$-dimensional complete intersections subordinate to Δ.*

Proof. Theorem 6.2.23 proves the required fact for $(2n-2)$-dimensional homology. In particular, it yields the representation for the cycles $T(\Gamma_i)$ corresponding to the facets $\Gamma_i \subset \Delta$. Each face Γ of dimension $n-m$ is the intersection of m facets $\Gamma_{i_1}, \dots, \Gamma_{i_m}$. The corresponding cycle $T(\Gamma)$ is the intersection of the cycles $T(\Gamma_{i_1}), \dots, T(\Gamma_{i_m})$, and so it can also be represented as specified. Since each cycle in $H_{2n-2m}(M_{\Delta^\perp}, \mathbb{Z})$ is a linear combination of cycles corresponding to $(n-m)$-dimensional faces, this completes the proof. □

6.3. Rings of conditions and rings of intersections as algebras with Gorenstein duality. A smooth projective toric variety $M_{\Delta^\perp}$ is an oriented even-dimensional real manifold homotopically equivalent to a cell complex containing even-dimensional cells only. The cohomology ring $H^*(M_{\Delta^\perp}, \mathbb{R})$ of such a manifold possesses Gorenstein duality (see Example 6.1.4). Like any algebra with Gorenstein duality, such a ring can be described in two slightly different ways. The cohomology ring of $M_{\Delta^\perp}$ with integer coefficients has no torsion and can be recovered from the algebra $H^*(M, \mathbb{R})$. We describe the rings $H^*(M_{\Delta^\perp}, \mathbb{R})$ and $H^*(M_{\Delta^\perp}, \mathbb{Z})$ in § 6.3.1.

Recall that the ring of conditions $\mathscr{R}_n$ of the complex torus $(\mathbb{C}^*)^n$ is an infinite-dimensional graded commutative algebra whose mth component consists of *algebraic cycles* of codimension m, that is, linear combinations of $(n-m)$-dimensional algebraic subvarieties of $(\mathbb{C}^*)^n$ defined up to *numerical equivalence.*

6.3.1. *The rings of intersections of toric varieties.* Let $\mathscr{G}_\Delta$ denote the Grothendieck group of the semigroup of lattice polytopes subordinate to the fan $\Delta^\perp$ for a fixed simple lattice polytope $\Delta \subset \mathbb{R}^n$. Let $\mathscr{L}_\Delta$ denote the finite-dimensional $\mathbb{R}$-linear space spanned by the virtual polytopes in $\mathscr{G}_\Delta$, so that $\mathscr{L}_\Delta = \mathscr{G}_\Delta \otimes_{\mathbb{Z}} \mathbb{R}$.

The space of $\mathbb{R}$-linear combinations of k-tuples of polytopes from $\mathscr{L}_\Delta$ can be identified in a natural way with the component $\mathscr{S}_k(\mathscr{L}_\Delta)$ of the symmetric algebra of $\mathscr{L}_\Delta$.

It will be convenient to deal with the ring of intersections $H_*(M_{\Delta^\perp}, \mathbb{R})$ dual to the cohomology ring $H^*(M_{\Delta^\perp}, \mathbb{R})$. We modify the grading on $H_*(M_{\Delta^\perp}, \mathbb{R})$ by ascribing grading k to the component $H_{n-2k}(M_{\Delta^\perp}, \mathbb{R})$. Let $H_{*'}(M_{\Delta^\perp}, \mathbb{R})$ denote the homology ring $H_*(M_{\Delta^\perp}, \mathbb{R})$ with the modified grading.

One description of $H_{*'}(M_{\Delta^\perp}, \mathbb{R})$ as an algebra with Gorenstein duality is this. Consider the n-form B on $\mathscr{L}_\Delta$ that assigns the mixed volume $\mathrm{MV}(\Delta_1, \dots, \Delta_n)$ times $n!$ to an n-tuple of virtual polytopes $\Delta_1, \dots, \Delta_n \in \mathscr{L}_\Delta$. The form B corresponds to a linear function $\mathscr{L}(B)$ on the symmetric algebra $\mathscr{S}(\mathscr{L}_\Delta)$ and to a homogeneous ideal $I_{\mathscr{L}(B)} \subset \mathscr{S}(\mathscr{L}_\Delta)$.

In the ideal $I_{\mathscr{L}(B)}$, its homogeneous component of degree k consists of linear combinations $\sum \lambda_i(\Delta_1^i, \dots, \Delta_k^i)$ of k-tuples $(\Delta_1^i, \dots, \Delta_k^i)$ of virtual polytopes in $\mathscr{L}_\Delta$ such that for each polytope $\widetilde{\Delta} \in \mathscr{L}_\Delta$ the linear combination

$$\sum \lambda_i n!\, \mathrm{MV}(\Delta_1^i, \dots, \Delta_k^i, \Delta_{k+1}, \dots, \Delta_n)$$

of mixed volumes of the polytopes $(\Delta_1^i, \dots, \Delta_k^i, \Delta_{k+1}, \dots, \Delta_n)$ with $\Delta_{k+1} = \dots = \Delta_n = \widetilde{\Delta}$ is equal to zero.

Theorem 6.3.1. *The ring $H_{*'}(M_{\Delta^\perp}, \mathbb{R})$ is isomorphic to the quotient algebra*

$$\mathscr{S}(\mathscr{L}_\Delta)_{\mathscr{L}(B)} = \mathscr{S}(\mathscr{L}_\Delta)/I_{\mathscr{L}(B)}$$

of the symmetric algebra $\mathscr{S}(\mathscr{L}_\Delta)$ of $\mathscr{L}_\Delta$ with respect to the ideal $I_{\mathscr{L}(B)}$.

Corollary 6.3.2. *The ring $H_{*'}(M_{\Delta^\perp}, \mathbb{Z})$ is isomorphic to the subring of $\mathscr{S}(\mathscr{L}_\Delta)_{\mathscr{L}(B)}$ generated by the images of the subring $\mathbb{Z}$ and the subgroup $\mathscr{G}_\Delta$ of the algebra $\mathscr{S}(\mathscr{L}_\Delta)$ under the factorization homomorphism.*

Now we give another description (called the Pukhlikov–Khovanskii description) of the ring $H_{*'}(M_{\Delta^\perp}, \mathbb{R})$ as an algebra with Gorenstein duality. Consider a homogeneous polynomial P of degree n on $\mathscr{L}_\Delta$ that assigns to a virtual polytope in $\mathscr{L}_\Delta$ the volume of the polytope. The polynomial P corresponds to an ideal $I_{\mathscr{L}(P)}$ of the algebra $\mathrm{Diff}(\mathscr{L}_\Delta)$ of linear differential operators with constant coefficients on $\mathscr{L}_\Delta$, which is isomorphic to $\mathscr{S}(\mathscr{L}_\Delta)$. The ideal $I_{\mathscr{L}(P)}$ consists of all operators vanishing at P.

Theorem 6.3.3. *The ring $H_{*'}(M_{\Delta^\perp}, \mathbb{R})$ is isomorphic to the quotient algebra*

$$\mathrm{Diff}(\mathscr{L}_\Delta)_{\mathscr{L}(P)} = \mathrm{Diff}(\mathscr{L}_\Delta)/I_{\mathscr{L}(P)}$$

of $\mathrm{Diff}(\mathscr{L}_\Delta)$.

Corollary 6.3.4. *The ring $H_{*'}(M_{\Delta^\perp}, \mathbb{Z})$ is isomorphic to the subring of* $\mathrm{Diff}(\mathscr{L}_\Delta)_{\mathscr{L}(P)}$ *generated by the images under the factorization homomorphism of the operators of multiplication by integers and the operators of differentiation along vectors in $\mathscr{G}_\Delta$.*

6.3.2. *The ring of conditions of* $(\mathbb{C}^*)^n$. Let $\mathscr{G}$ denote the Grothendieck group of the semigroup of lattice polytopes in $\mathbb{R}^n$. Let $\mathscr{L}$ be the finite-dimensional $\mathbb{R}$-linear space spanned by the virtual polytopes in $\mathscr{G}$, that is, $\mathscr{L} = \mathscr{G} \otimes_{\mathbb{Z}} \mathbb{R}$.

The space of $\mathbb{R}$-linear combinations of k-tuples of polytopes from $\mathscr{L}$ can be identified in a natural way with the component $\mathscr{S}_k$ of the symmetric algebra of $\mathscr{L}$.

One description of the ring of conditions $\mathscr{R}_n$ is this. Consider the n-form B on $\mathscr{L}$ that assigns their mixed volume $\mathrm{MV}(\Delta_1, \dots, \Delta_n)$ times $n!$ to an n-tuple of virtual polytopes $\Delta_1, \dots, \Delta_n \in \mathscr{L}$. The form B corresponds to a linear function $\mathscr{L}(B)$ on the symmetric algebra $\mathscr{S}(\mathscr{L})$ and a homogeneous ideal $I_{\mathscr{L}(B)} \subset \mathscr{S}(\mathscr{L})$.

The homogeneous component of $I_{\mathscr{L}(B)}$ of degree k consists of linear combinations $\sum \lambda_i(\Delta_1^i, \dots, \Delta_k^i)$ of k-tuples $(\Delta_1^i, \dots, \Delta_k^i)$ of virtual polytopes in $\mathscr{L}$ such that for each polytope $\widetilde{\Delta} \in \mathscr{L}$ the linear combination

$$\sum \lambda_i n!\, \mathrm{MV}(\Delta_1^i, \dots, \Delta_k^i, \Delta_{k+1}, \dots, \Delta_n)$$

of the mixed volumes of the tuples $(\Delta_1^i, \dots, \Delta_k^i, \Delta_{k+1}, \dots, \Delta_n)$ with $\Delta_{k+1} = \dots = \Delta_n = \widetilde{\Delta}$ is equal to zero.

Theorem 6.3.5. *The ring $\mathscr{R}_n$ is isomorphic to the subring of $\mathscr{S}(\mathscr{L})/I_{\mathscr{L}(B)}$ generated by the images of the subring $\mathbb{Z}$ and the subgroup $\mathscr{G}$ of $\mathscr{S}(\mathscr{L})$ under the factorization homomorphism.*

The other description of the ring of conditions $\mathscr{R}_n$ is as follows. Consider the homogeneous polynomial P of degree n on $\mathscr{L}$ that assigns to a virtual polytope in $\mathscr{L}$ the volume of this polytope. The polynomial P corresponds to the ideal $I_{\mathscr{L}(P)}$ in the algebra $\mathrm{Diff}(\mathscr{L})$ of linear differential operators with constant coefficients on $\mathscr{L}$ (which is isomorphic to $\mathscr{S}(\mathscr{L})$). The ideal $I_{\mathscr{L}(P)}$ consists of the operators vanishing at P.

Theorem 6.3.6. *The ring $\mathscr{R}_n$ is isomorphic to the subring of $\mathrm{Diff}(\mathscr{L})/I_{\mathscr{L}(P)}$ generated by the images under the factorization homomorphism of the operators of multiplication by integers and the operators of differentiation along vectors in $\mathscr{G}$.*

7. The ring of conditions of $\mathbb{C}^n$

7.1. Introduction. The algorithm for constructing the ring of conditions of a homogeneous affine algebraic variety X terminates successfully if the ring of polynomials on X has a 'nice structure' from the standpoint of group theory; see [11] and [10]. It is usually assumed that 1) the acting group H is reductive, and 2) the representation of H in the ring of polynomials on X contains no multiple irreducible components. For example, when $H = X = (\mathbb{C} \setminus \{0\})^n$, conditions 1) and 2) are obviously satisfied because in this case the ring of polynomials consists of linear combinations of characters of the torus $(\mathbb{C} \setminus \{0\})^n$.

The ring of exponential sums consists of linear combinations of characters of the additive group $\mathbb{C}^n_+$ of $\mathbb{C}^n$. Recall that an *exponential sum* is a function of the form

$$f(z) = \sum_{\lambda \in \Lambda \subset (\mathbb{C}^n)^*,\ c_\lambda \in \mathbb{C}} c_\lambda \mathrm{e}^{\langle z, \lambda \rangle}$$

on $\mathbb{C}^n$, where Λ is a finite subset of the dual space $(\mathbb{C}^n)^*$ of $\mathbb{C}^n$, called the *support* of the exponential sum. An exponential sum with support in the subspace $\mathrm{Re}(\mathbb{C}^n)^* \subset (\mathbb{C}^n)^*$ is said to be *quasi-algebraic*. The convex hull of the support of an exponential sum is called its *Newton polytope*.

We consider the ring of exponential sums as an analogue of the ring of Laurent polynomials on the complex torus. Guided by this analogy, we construct the ring of conditions of the corresponding intersection theory. More precisely, we consider *exponential analytic sets* (EA-sets in what follows), which are the varieties defined as the sets of common zeros of finite systems of exponential sums, and we construct the *ring of conditions* of the intersection theory of EA-sets. With an EA-set we associate an algebraic subvariety of a multidimensional complex torus, which we call a *model* of this EA-set; see Definition 7.2.2. If we limit ourselves to quasi-algebraic exponential sums, then the construction of the ring of conditions is based entirely on applying the methods of tropical algebraic geometry to models of EA-sets. For arbitrary exponential sums the geometry of EA-sets becomes more involved. In particular, in the general case, apart from standard tropical geometry, a certain complex extension of tropical concepts (see [31]) is used. For this reason, in the framework of our survey *we consider only quasi-algebraic* exponential sums and EA-sets. We state the main results and provide precise descriptions of the required constructions, but we omit the proofs, with a few exceptions.

Let $G \subset \mathrm{Re}(\mathbb{C}^n)^*_+$ be a subgroup with finitely many generators. Assume that G contains a basis of $(\mathbb{C}^n)^*$. Let E_G denote the ring of exponential sums with support in G. Next we construct an intersection theory for EA-sets with equations in E_G. To do this we determine the corresponding ring of conditions $\mathscr{E}_G$. The ring of exponential sums is the direct limit of the rings E_G over all subgroups $G \subset \mathrm{Re}(\mathbb{C}^n)^*_+$. Correspondingly, the ring of conditions $\mathscr{E}$ of all EA-sets is the direct limit of the graded commutative $\mathbb{Z}$-algebras $\mathscr{E}_G$. In addition, $\mathscr{E}$ has the structure of a graded $\mathbb{R}$-algebra.

In the definition of the ring of conditions we use the intersection number $I(X, Y)$ of two EA-sets X and Y of total dimension n. To construct $I(X, Y)$ we define the concepts of algebraic codimension $\operatorname{codim}_{\mathrm{a}} X$ and equidimensionality of an EA-set X; see Definition 7.2.3. The algebraic codimension usually coincides with the codimension of X as an analytic subset of $\mathbb{C}^n$ (see Example 7.2.4). Equidimensionality is a substitute for the concept of irreducibility. Namely, each EA-set can be represented uniquely as a finite union of equidimensional EA-sets of differing algebraic codimensions.

Next we introduce the concept of *weak density* $d_{\mathrm{w}}(X)$ of an equidimensional EA-set X of algebraic codimension n; see Definition 7.2.12. The set X is infinite. (For instance, for $n = 1$ it is the set of zeros of $\mathrm{e}^z - 1$.) The weak density of X is an analogue of the cardinality of a zero-dimensional algebraic variety. It turns out that if $\operatorname{codim}_{\mathrm{a}} X + \operatorname{codim}_{\mathrm{a}} Y = n$, then there exists a *domain of relatively full measure* $U \subset \mathrm{Re}\,\mathbb{C}^n$ (see Definition 7.2.7) such that $d_{\mathrm{w}}(X \cap (z + Y))$ is constant when $\mathrm{Re}\, z \in U$. We put

$$I(X, Y) = d_{\mathrm{w}}(X \cap (z + Y)),$$

and call it the intersection number of the EA-sets X and Y. The quantity I is symmetric and invariant under the action $z\colon X \mapsto z + X$ of the additive group $\mathbb{C}^n_+$ of $\mathbb{C}^n$. In the definitions of weak density and the intersection number we use the group G such that the equations of X and Y belong to E_G. The values $d_{\mathrm{w}}(X)$ and $I(X, Y)$ belong to a certain subgroup of $\mathbb{R}_+$. The union of these subgroups over all subgroups $G \subset \mathrm{Re}(\mathbb{C}^n)^*_+$ is the whole of $\mathbb{R}$. In addition, $d_{\mathrm{w}}(X)$ and $I(X, Y)$ are independent of the choice of G for any fixed X and Y.

The definition of the ring of conditions is modelled on [11] and [10]. We say that two equidimensional EA-sets X and Y with algebraic codimensions $k \leqslant n$ are equivalent if $I(X, Z) = I(Y, Z)$ for each equidimensional EA-set Z of algebraic codimension $n - k$. All EA-sets of algebraic codimension $> n$ are also said to be equivalent. The sets of equivalence classes of algebraic codimension k form a homogeneous component of the graded commutative semiring with operations defined as follows.

Fix the equivalence classes containing the equidimensional EA-sets X and Y. Then there exists a domain $U \subset \mathrm{Re}\,\mathbb{C}^n$ of relatively full measure, which depends on X and Y, such that the following conditions hold for any $z \in U + \mathrm{Im}\,\mathbb{C}^n$:

1) the EA-sets $X\cap(z+Y)$ are equidimensional;

2) their equivalence classes are equal and do not depend on the choice of X or Y;

3) if $\operatorname{codim}_{\mathrm{a}} X=\operatorname{codim}_{\mathrm{a}} Y$, then the equivalence classes of the EA-sets $X\cup(z+Y)$ are equal.

Let $\iota(Z)$ denote the equivalence class of an EA-set Z. Taking an arbitrary $z\in U+\operatorname{Im}\mathbb{C}^n$, we put

$$\iota(X)+\iota(Y)=\iota(X\cup(z+Y)) \quad \text{and} \quad \iota(X)\cdot\iota(Y)=\iota(X\cap(z+Y)).$$

We use tropical techniques to verify that the operations of addition and multiplication are well defined; see § 7.3. Let a, b, and c be equivalence classes in the same codimension. Then, by definition, the equality $a+c=b+c$ implies that $a=b$. Consider the Grothendieck group $\mathscr{E}_{G,k}$ of the semigroup of equivalence classes of codimension k, and let $\mathscr{E}_G=\mathscr{E}_{G,0}+\mathscr{E}_{G,1}+\cdots+\mathscr{E}_{G,n}$ denote the corresponding commutative graded $\mathbb{Z}$-algebra. We say that EA-sets with the same image in the ring of conditions are *numerically equivalent.* Below we list the main properties of the ring of conditions $\mathscr{E}_G=\mathscr{E}_{G,0}+\cdots+\mathscr{E}_{G,n}$:

(R1) $\mathscr{E}_{G,0}=\mathbb{Z}$;

(R2) the weak density of an EA-set is constant on numerical equivalence classes; there exists a $\varpi(G)\in\mathbb{Z}$ such that the map $d_{\mathrm{w}}\colon \mathscr{E}_{G,n}\to\varpi(G)\mathbb{Z}$ is a group isomorphism;

(R3) when $p+q=n$, multiplication defines a non-degenerate pairing $\mathscr{E}_{G,p}\times\mathscr{E}_{G,q}\to\mathscr{E}_{G,n}\xrightarrow{d_{\mathrm{w}}}\varpi(G)\mathbb{Z}$;

(R4) the algebra $\mathscr{E}_G\otimes_{\mathbb{Z}}\mathbb{Q}$ is generated by the elements $\mathscr{E}_{G,1}\otimes_{\mathbb{Z}}\mathbb{Q}$ (hence the ring of conditions $\mathscr{E}$ is generated by the images of exponential hypersurfaces);

(R5) when $G=\mathbb{Z}^n$, $\mathscr{E}_G$ is the ring of conditions of the complex torus $(\mathbb{C}\setminus\{0\})^n$.

The structure of the ring of conditions of the complex torus can be described in various ways in terms of the geometry of polytopes with vertices at integer lattice points (Newton polytopes); see §§ 4.2.1 and 6.3.1 in this survey, and also [7], [32], and [31]. These descriptions remain valid after passing to the ring of conditions $\mathscr{E}$, provided that we drop the condition that the polytopes should be lattice polytopes. We present one such description in § 7.3 (Theorem 7.3.13).

By definition, if $X_1,\dots,X_k$ are EA-sets such that the sum of their algebraic codimensions is n, then their intersection number is $d_{\mathrm{w}}(\overline{X}_1\cdots\overline{X}_k)$, where $\overline{X}_i$ is the image of the EA-set X_i in the ring of conditions (cf. Definition 7.2.15). When $k=n$, let X_i be a hypersurface with equation $f_i=0$. Then, in the quasi-algebraic case, the intersection number of the divisors $X_1,\dots,X_n$ is equal to the mixed volume of the Newton polyhedra of the EA-set f_i. This can be viewed as an analogue of the Kushnirenko–Bernstein formula (also called the BKK or Bernstein–Kushnirenko–Khovanskii formula) for the number of solutions of a polynomial system of equations; see Theorem 2.2.1. It refines the previously known results [27], [38].

7.2. Densities and intersections of EA-sets.

7.2.1. *Models and windings.* To define the weak density of an EA-set (see Definition 7.2.12) we regard it as the intersection of an algebraic variety, called a *model* of the EA-set, with the multidimensional winding on the complex torus; see Definitions 7.2.1 and 7.2.2.

Let $G \subset \mathrm{Re}(\mathbb{C}^n)^*_+$ be a subgroup with a finite number of generators. Assume that G contains a basis of the space $\mathrm{Re}(\mathbb{C}^n)^*$. Let E_G denote the ring of exponential sums with support in G. For any $z \in \mathbb{C}^n$ we take $\omega(z)$ to be the character of G defined by $\omega(z)\colon g \mapsto \mathrm{e}^{\langle z,g\rangle}$. Let $\mathbb{T}$ be the torus of characters of G. In this way we obtain an embedding of groups $\omega\colon \mathbb{C}^n_+ \to \mathbb{T}$.

Definition 7.2.1. We call the image $\omega(\mathbb{C}^n)$ of ω the standard winding on the torus $\mathbb{T}$, and $\omega\colon \mathbb{C}^n \to \mathbb{T}$ the standard winding map.

The standard winding on the torus is dense in the Zariski topology. Hence $\omega^*\colon \mathbb{C}[\mathbb{T}] \to E_G$ is a ring isomorphism.[2]

Definition 7.2.2. Let I be the ideal of E_G generated by the equations of an EA-set X. Let $\kappa(X) \subset \mathbb{T}$ denote the zero locus of the ideal $(\omega^*)^{-1}I \subset \mathbb{C}[\mathbb{T}]$. We call $\kappa(X)$ a model of the EA-set X.

The EA-set X coincides with $\omega^{-1}\kappa(X)$. Conversely, to an arbitrary algebraic variety $M \subset \mathbb{T}$ there corresponds the EA-set $\omega^{-1}M$. We have $M = \kappa(\omega^{-1}M)$, so that M is a model of the EA-set $\omega^{-1}M$. Calculating the ring $\mathscr{E}_G$ is based on the fact that the map $M \mapsto \omega^{-1}M$ extends to a homomorphism of the ring of conditions of the torus $\mathbb{T}$ onto $\mathscr{E}_G$; see § 7.3.

Definition 7.2.3. Setting $\mathrm{codim_a}\, X = \mathrm{codim}\, \kappa(X)$, we call $\mathrm{codim_a}\, X$ the algebraic codimension of the EA-set X. If the variety $\kappa(X)$ is equidimensional (that is, it is made up of irreducible components of equal dimension), then X is also said to be equidimensional.

Increasing the group G preserves codimension and the property of equidimensionality of the model $\kappa(X)$ for a fixed EA-set X (by contrast to dimension and the property of irreducibility). Each EA-set is a union of a finite number of equidimensional EA-sets of differing algebraic codimensions. In what follows we assume by default that each EA-set under consideration is equidimensional.

Example 7.2.4 (see [28], [68], and [6]). Assume that the EA-set X is given by equations $f = g = 0$. If f and g have no common divisor in E_G, then $\mathrm{codim_a}\, X = 2$; otherwise $\mathrm{codim_a}\, X = 1$. In particular, $0 \in \mathbb{C}$ treated as the EA-set given by the equations $\mathrm{e}^z - 1 = \mathrm{e}^{\sqrt{2}\,z} - 1 = 0$ has algebraic codimension 2. Thus, the codimension of an analytic set X can be lower than $\mathrm{codim_a}\, X$. Let (X, z) be the irreducible germ of an EA-set X at $z \in X$. If (X, z) has lower codimension than $\mathrm{codim_a}\, X$, then the germ is said to be *atypical*. It is known that each atypical germ of an EA-set lies in a proper affine subspace of $\mathbb{C}^n$. In particular, each atypical component of an EA-set of algebraic codimension 2 in $\mathbb{C}^2$ is an affine line. In addition, it is known that the set of minimal affine subspaces containing atypical components is small in a certain sense.

Let X be an EA-set given by equations in the ring E_G, let $\mathbb{T}$ be the character torus of the group G, and let $\kappa(X) \subset \mathbb{T}$ be a model of the EA-set X. For any $t \in \mathbb{T}$ we write $X_t = \omega^{-1}(t\kappa(X))$. The variety $t\kappa(X) \subset \mathbb{T}$ is a model of the EA-set X_t.

[2] The practice of viewing exponential sums as restrictions of Laurent polynomials to a dense winding on the torus goes back to Weyl's celebrated paper [67].

In this way we obtain an action of $\mathbb{T}$ on EA-sets defined as the zero varieties of systems of exponential sums in the ring E_G. We describe this action in detail.

Fix a basis $\alpha_1, \dots, \alpha_N$ of G. Then each exponential sum $f \in E_G$ can be expressed uniquely as a Laurent polynomial

$$f(z) = P(e^{\alpha_1(z)}, \dots, e^{\alpha_N(z)})$$

of the variables $e^{\alpha_1(z)}, \dots, e^{\alpha_N(z)}$. When $t = (c_1, \dots, c_N) \in (\mathbb{C}\setminus\{0\})^N$ (identifying $\mathbb{T}$ and $(\mathbb{C} \setminus \{0\})^N$), we put

$$f(z) \xrightarrow{t} tf(z) = P(c_1 e^{\alpha_1(z)}, \dots, c_N e^{\alpha_N(z)}).$$

If the EA-set is given by equations $f_1 = \dots = f_k = 0$, then the action of t takes it to the EA-set given by $tf_1 = \dots = tf_k = 0$. This action of $\mathbb{T}$ is an extension of the shift action of $\mathbb{C}^n$ on exponential sums and EA-sets. Using just this action of $\mathbb{T}$, we can consistently define the weak density of an EA-set of codimension n and then the intersection number $I(X, Y)$ of two EA-sets X and Y of complementary codimension; see Theorem 7.2.11 and Definition 7.2.15 below. The definitions of weak density and the intersection number use the given group G. However, it follows from Theorems 7.2.14 and 7.3.1 that these quantities are independent of the choice of this group. In what follows we assume that $G \subset \mathrm{Re}(\mathbb{C}^n)^*$ is fixed.

Below we use the following *convention on the multiplicity of points* in EA-sets. Let $M = \bigcup_i m_i M_i$, where the M_i are the irreducible components of a model M of an EA-set X of algebraic codimension n, and let $z \in X$. We call z a *normal point* in X if there exists a k such that

(i) $\omega(z) \subset M_k \setminus \bigcup_{i \neq k} M_i$;

(ii) the point $\omega(z) \in M_k$ is non-singular;

(iii) the standard winding intersects M_k transversally at $\omega(z)$.

We assign multiplicity m_k to the normal point z. The property of $x \in X$ being a normal point and the value of its multiplicity are independent of the choice of a model M of the EA-set X. If we assign fixed multiplicity K to all isolated points in X that are not normal, then the weak densities and intersection numbers (to be defined below) of the EA-set will be independent of the choice of K. For completeness, we assign multiplicity zero to all isolated points of the EA-set that are not normal.

7.2.2. *The density and intersection number of EA-sets.* We need the following definitions.

Definition 7.2.5. Let B_r be the ball with radius r and centre at the origin in a finite-dimensional Euclidean space E, let σ_n be the volume of an n-dimensional ball with radius 1, and let $Y \subset E$ be a discrete set of points with non-negative multiplicity. Let $N(Y, r)$ be the cardinality of $Y \cap B_r$. If the limit

$$\lim_{r \to \infty} \frac{N(Y, r)}{\sigma_n r^n}$$

exists, then we call it the n-density of Y and denote it by $d_n(Y)$.

Note that the n-density of a set depends on the choice of a metric in E.

Example 7.2.6. If $X \subset \mathbb{C}$ is an EA-set given by an equation $f(z) = 0$, then the 1-density $d_1(X)$ exists and is equal to $p/(2\pi)$, where p is the perimeter of the Newton segment of the exponential sum f.

Definition 7.2.7. Let $\mathfrak{I} = \{I\}$ be a finite set of proper subspaces of a real vector space E. Put

$$B_{\mathfrak{I}} = E \setminus \bigcup_{I \in \mathfrak{I}} I$$

and let $B_{\mathfrak{I},1}, B_{\mathfrak{I},2}, \dots$ be the connected components of $B_{\mathfrak{I}}$. When $0 < R \in \mathbb{R}$, let $B_{\mathfrak{I}}^R$ denote the subset of E consisting of the points lying at a distance $\geqslant R$ from $\bigcup_{I \in \mathfrak{I}} I$. We call a domain $U \subset E$ that contains a subdomain of the form $B_{\mathfrak{I}}^R$ a domain of relatively full measure (an RFD in what follows) with base $\bigcup_{I \in \mathfrak{I}} I$.

Here are some consequences of Definition 7.2.7.

Corollary 7.2.8. 1) *Unions and intersections of RFDs are RFDs.*

2) *The fact that a domain is an RFD is independent of the choice of metric in E.*

3) *If a subspace $L \subset E$ does not lie in the base of an RFD U, then $U \cap L$ is an RFD in L.*

Definition 7.2.9. 1) Let $\mathscr{L} \subset E$ be a lattice in E with positive integer multiplicity $m(\mathscr{L})$, and let $X \subset E$ be a set of points with multiplicities. A set $X \subset E$ is called an ε-perturbation of the translated lattice $z + \mathscr{L}$ if (a) X lies in the ε-neighbourhood $(z + \mathscr{L})_\varepsilon$ of this translated lattice, and (b) the ε-neighbourhood of each point $x \in z + \mathscr{L}$ contains precisely $m(\mathscr{L})$ points from X.

2) If $X_1, \dots, X_m$ are ε-perturbations of translated lattices $z_j + \mathscr{L}_j$, then

$$\bigcup_{1 \leqslant j \leqslant m} X_j$$

is called an ε-perturbation of the union of translated lattices $\bigcup_{1 \leqslant j \leqslant m} (z_j + \mathscr{L}_j)$.

Corollary 7.2.10. *Let X be an ε-perturbation of the union of translated lattices $\{z_j + \mathscr{L}_j\}$. If $\operatorname{rk} \mathscr{L}_j = n$ for each j, then the n-density $d_n(X)$ exists and is equal to $\sum_j d_n(\mathscr{L}_j)$.*

Below we claim that:

(i) (Theorem 7.2.11) for 'almost all sufficiently large' $t \in \mathbb{T}$ the shifts tX of a quasi-algebraic EA-set X of algebraic codimension n can be approximated by unions of translations of lattices in a fixed finite set of sublattices in the space $\operatorname{Im} \mathbb{C}^n$; the n-densities of the EA-sets tX are defined and equal to one another;

(ii) (Theorem 7.2.14) if X and Y are quasi-algebraic EA-sets such that the sum of their algebraic codimensions is n, then for 'almost all sufficiently large' $z \in \mathbb{C}^n$ the EA-sets $(z+X) \cap Y$ are equidimensional; they all have algebraic codimension n and the same weak density.

In what follows we use this notation: $\mathfrak{T}$ is the Lie algebra of the torus $\mathbb{T}$, $\operatorname{Re}\mathfrak{T}$ and $\operatorname{Im}\mathfrak{T}$ are the real and imaginary subspaces of $\mathfrak{T}$, and $\exp\colon \mathfrak{T}\to\mathbb{T}$ is the exponential map.

Theorem 7.2.11. *Let* $\operatorname{codim_a} X = n$. *Then there is a set of subspaces* $\mathfrak{J}=\{I\subset \operatorname{Re}\mathfrak{T}\}$ *and there are finite systems of full rank lattices* $\{\mathscr{L}_{i,j}\subset \operatorname{Im}\mathbb{C}^n \mid j=1,2,\dots\}$ *corresponding to the connected components* $B_{\mathfrak{J},i}$ (*see Definition* 7.2.7) *such that*

1) *for each* ε *there exists an* $R>0$ *such that if* $t\in\exp(B^R_{\mathfrak{J},i}+\operatorname{Im}\mathfrak{T})$, *then the EA-set* X_t *is an* ε*-perturbation of the union of translated lattices* $z_1(t)+\mathscr{L}_{i,1}, z_2(t)+\mathscr{L}_{i,2},\dots$, *where the functions* $z_j(t)$ *are continuous*;

2) *the* n*-density* $d_n(X_t)$ *is independent of the choice of a connected component* $B_{\mathfrak{J},i}$ *containing* $\operatorname{Re}\log t$.

Definition 7.2.12. When $t\in\exp(B^R_{\mathfrak{J},i}+\operatorname{Im}\mathfrak{T})$, we put $d_{\mathrm{w}}(X)=d_n(X_t)$ and call $d_{\mathrm{w}}(X)$ the weak density of X. If $d_n(X)$ exists and is equal to $d_{\mathrm{w}}(X)$, then we call $d_n(X)$ the density of the EA-set X.

Corollary 7.2.13. *The density and weak density of an* EA*-set are preserved under the shift action of* $\mathbb{C}^n$, *that is*, $d_{\mathrm{w}}(X)=d_{\mathrm{w}}(z+X)$.

Theorem 7.2.14. *Let* $\operatorname{codim_a} X+\operatorname{codim_a} Y=n$. *Then there exists* $\mathfrak{J}=\{I\subset \operatorname{Re}\mathbb{C}^n\}$ *such that if* R *is sufficiently large*, *then the following conditions hold for all* $z\in B^R_{\mathfrak{J}}+\operatorname{Im}\mathbb{C}^n$:

(i) *the* EA*-sets* $(z+X)\cap Y$ *are equidimensional*;

(ii) $\operatorname{codim_a}((z+X)\cap Y)=n$;

(iii) *the weak densities* $d_{\mathrm{w}}((z+X)\cap Y)$ *are equal to one another.*

Definition 7.2.15 (the intersection number of EA-sets). Let $B^R_{\mathfrak{J}}\subset\operatorname{Re}\mathbb{C}^n$ be the RFD from Theorem 7.2.14. Then we put

$$I(X,Y)=d_{\mathrm{w}}\bigl((z+X)\cap Y\bigr)\quad\text{for}\quad z\in B^R_{\mathfrak{J}}+\operatorname{Im}\mathbb{C}^n.$$

Corollary 7.2.16. *For any* $z,w\in\mathbb{C}^n$

$$I(z+X,w+Y)=I(X,Y).$$

Recall that, by definition, the intersection number $I(X_1,\dots,X_n)$ of exponential hypersurfaces $X_1,\dots,X_n$ is $d_{\mathrm{w}}(\overline{X}_1\cdots\overline{X}_n)$, where $\overline{X}_i$ is the image of the EA-set X_i in the ring of conditions (cf. Definition 7.2.15).

Theorem 7.2.17. *Let* $X_i=\{z\in\mathbb{C}^n\colon f_i(z)=0,\ i=1,\dots,n\}$ *be exponential hypersurfaces. Then*

$$I(X_1,\dots,X_n)=\frac{n!}{(2\pi)^n}\operatorname{MV}(\Delta_1,\dots,\Delta_n),$$

where $\operatorname{MV}(\Delta_1,\dots,\Delta_n)$ *is the mixed volume of the Newton polytopes of the exponential sums* f_i.

7.3. EA-sets and tropical algebraic geometry. In this subsection we explain the relationship between the notions of weak density and intersection number for EA-sets and tropical algebraic geometry. It turns out that these quantities depend only on the tropicalizations of models of EA-sets. We present a surjective map of the ring of conditions of the complex torus onto the quasi-algebraic ring of conditions $\mathscr{E}_G$ and establish a connection between $\mathscr{E}_G$ and the ring of convex polytopes. We gave all necessary tropical definitions in § 4.2.

Let H be an n-dimensional subspace of $\operatorname{Re}\mathfrak{T}$. Fix a Euclidean metric in H. In what follows we construct a linear functional $\mathscr{Y}$ on the space of tropical fans of codimension n in $\operatorname{Re}\mathfrak{T}$. The value of this functional on a tropical fan $\mathscr{K}$ can be viewed as an 'intersection number' of the fan and H. We present the definition of $\mathscr{Y}$ immediately after the following statements.

Theorem 7.3.1. *Let X be an EA-set given by equations in the ring E_G, let $\operatorname{codim}_{\mathrm{a}} X = n$, let $\mathbb{T}$ be the character torus of G, and let $\operatorname{trop}(X) \subset \operatorname{Re}\mathfrak{T}$ be the tropicalization of a model of the EA-set X. We put $H = d\omega(\operatorname{Re}\mathbb{C}^n) \subset \operatorname{Re}\mathfrak{T}$, where ω is the standard winding map (see Definition 7.2.1). Fix the Euclidean metric in H induced by the metric in $\operatorname{Re}\mathbb{C}^n$. Then*

$$d_{\mathrm{w}}(X) = \mathscr{Y}(\operatorname{trop}(X)).$$

Corollary 7.3.2. *If the models of* EA*-sets X and Y have the same tropicalization, then $d_{\mathrm{w}}(X) = d_{\mathrm{w}}(Y)$.*

Theorem 7.3.3. *Let $\operatorname{codim}_{\mathrm{a}} X + \operatorname{codim}_{\mathrm{a}} Y = n$. Then*

$$I(X,Y) = \mathscr{Y}(\operatorname{trop}(X)\cdot\operatorname{trop}(Y)),$$

where $\operatorname{trop}(X)\cdot\operatorname{trop}(Y)$ is the product of tropical fans (defined in § 4.2.1).

Corollary 7.3.4. *If the models of the EA-sets X and Y have the same tropicalization, then $I(X,Z) = I(Y,Z)$ for each* EA*-set Z.*

We turn to the definition of $\mathscr{Y}$.

Let $S \subset \operatorname{Re}\mathfrak{T}$ be a rational subspace (that is, spanned by vectors in the integer lattice) of codimension n. Consider the image $\pi(C)$ of the unit cube C in H under the projection $\pi\colon H \to \operatorname{Re}\mathfrak{T}/S$ and put $\eta(H,S) = \operatorname{Vol}(\pi(C))$, where the volume form in $\operatorname{Re}\mathfrak{T}/S$ is chosen in such a way that the fundamental cube in the integer quotient lattice has volume 1.

Let $\mathscr{K}$ be a tropical fan of codimension n in $\operatorname{Re}\mathfrak{T}$ and let $\operatorname{supp}\mathscr{K} = \bigcup_{K\in\mathscr{K}} K$. For any $K \in \mathscr{K}$, if the minimal subspace $\operatorname{Re}\mathfrak{T}$ containing H and K is proper, then we denote it by $I_{H,K}$. The set of subspaces of the form $I_{H,K}$ is denoted by $\mathfrak{I} = \{I\}$. Let $u \in B_{\mathfrak{I}}$ (see Definition 7.2.7). Then $H \cap (u + \operatorname{supp}\mathscr{K})$ is finite and consists of points of the form $H \cap (u+K)$, where $K \in \mathscr{K}$ is a cone of codimension n. For $u \in B_{\mathfrak{I}}$ we put

$$\mathscr{Y}_u(\mathscr{K}) = \sum_{K\in\mathscr{K},\, H\cap(u+K)\neq\varnothing} \eta(H,\mathfrak{T}_K)w(K),$$

where $\mathfrak{T}_K$ is the subspace of $\operatorname{Re}\mathfrak{T}$ spanned by K, and where $w(K)$ is the weight of K in the tropical fan $\mathscr{K}$. The value $\mathscr{Y}_u(\mathscr{K})$ of the functional is independent of the choice of $u \in B_{\mathfrak{I}}$. Thus, the functional $\mathscr{Y}$ is well defined.

The independence just mentioned can be explained as follows. Choose a rational subspace of $\operatorname{Re}\mathfrak{T}$ that is close to H, and introduce in it a Euclidean metric close to that on H. Then the claim reduces to the case of a rational subspace because $\mathscr{Y}_u(\mathscr{K})$ depends continuously on the Euclidean subspace H. Suppose that H is a rational space. We view it as a tropical fan with weight 1. Then $\mathscr{Y}_u(\mathscr{K})$ differs from the intersection number of the tropical varieties $\mathscr{K}$ and H by a constant factor (see Remark 4.2.2 in § 4.2.1). Therefore it is independent of the translation u.

Now we present a few known facts about the ring of conditions of the complex torus and tropicalizations of algebraic varieties. For this we use the following notation:

- $\mathscr{R}_{\mathbb{T}}$ is the ring of conditions of $\mathbb{T}$;
- $\mathbf{r}(M)$ is the numerical equivalence class of M;
- $\operatorname{Trop}_{\mathbb{T}}$ is the ring of tropical fans in $\operatorname{Re}\mathfrak{T}$;
- $\mathbf{trop}(M)$ is the tropicalization of M.

Assertion 7.3.5. *For any pair of varieties $M, N \subset \mathbb{T}$ there exists an algebraic hypersurface $D \subset \mathbb{T}$ such that for each $t \notin D$*

(i) $\mathbf{r}\big(M \cap (tN)\big) = \mathbf{r}(M) \cdot \mathbf{r}(N)$,

(ii) $\mathbf{trop}\big(M \cap (tN)\big) = \mathbf{trop}(M) \cdot \mathbf{trop}(N)$.

Assertion 7.3.6. *The map* **trop** *is constant on numerical equivalence classes and it is a ring isomorphism $\iota\colon \mathscr{R}_{\mathbb{T}} \to \operatorname{Trop}_{\mathbb{T}}$.*

Recall that the amoeba of an algebraic variety M is the image of M under the map $\operatorname{Re}\log\colon \mathbb{T} \to \operatorname{Re}\mathfrak{T}$.

Assertion 7.3.7 (for instance, see [30]). *The amoeba of an algebraic variety M lies at a finite distance from the Bergman cone* $\operatorname{supp}\mathbf{trop}(M)$ *of M.*

A significant step towards verifying that the ring of conditions $\mathscr{E}_G$ is well defined is proving that multiplication of numerical equivalence classes is well defined. The result we need is as follows.

Lemma 7.3.8. *Let X and Y be equidimensional* EA*-sets. Then there exists an* RFD *$U \subset \operatorname{Re}\mathbb{C}^n$ such that for each* EA*-set Z with algebraic codimension $n - \operatorname{codim}_{\mathrm{a}} X - \operatorname{codim}_{\mathrm{a}} Y$ the intersection numbers $I(X \cap (z+Y), Z)$ are the same for any $z \in U + \operatorname{Im}\mathbb{C}^n$.*

Proof. Let $\omega\colon \mathbb{C}^n \to \mathbb{T}$ be the standard winding map (see Definition 7.2.1) and let M, N, and P be models of the EA-sets X, Y, and Z, respectively. Then the variety $M \cap (\omega(z)N)$ is a model of the EA-set $X \cap (z+Y)$. From Theorem 7.3.3 we obtain

$$I\big(X \cap (z+Y), Z\big) = \mathscr{Y}\big(\mathbf{trop}\big(M \cap (\omega(z)N)\big) \cdot \mathbf{trop}(P)\big).$$

Consider the hypersurface D in Assertion 7.3.5 and let $\mathscr{I} = \{I \subset \operatorname{Re}\mathfrak{T}\}$ be a set of rational subspaces such that their union contains the support of the tropical fan of D. Since ω is a dense winding, it follows that $d\omega(\operatorname{Re}\mathbb{C}^n) \not\subset \bigcup_{I\in\mathscr{I}} I$. Therefore, $U = (d\omega)^{-1}(B^R_{\mathscr{I}})$ is an RFD; see Definition 7.2.7 and Corollary 7.2.8, 3). If R is sufficiently large, then $D \cap B^R_{\mathscr{J}} = \varnothing$ by Assertion 7.3.7. Hence $\omega(z) \notin D$ for each $z \in U + \operatorname{Im}\mathbb{C}^n$. Let $z \in U + \operatorname{Im}\mathbb{C}^n$. Then it follows from Assertion 7.3.5, (ii) that

$$\mathbf{trop}\bigl(M \cap (\omega(z)N)\bigr) = \mathbf{trop}(M) \cdot \mathbf{trop}(N).$$

Hence

$$I\bigl(X \cap (z+Y), Z\bigr) = \mathscr{Y}\bigl(\mathbf{trop}(M) \cdot \mathbf{trop}(N) \cdot \mathbf{trop}(P)\bigr), \tag{7.1}$$

that is, $I\bigl(X \cap (z+Y), Z\bigr)$ is independent of $z \in U + \operatorname{Im}\mathbb{C}^n$. □

For a model M of an EA-set X let M^{eas} denote the image of X in $\mathscr{E}_G$.

Theorem 7.3.9. *There exists a surjective ring homomorphism $s\colon \operatorname{Trop}_{\mathbb{T}} \to \mathscr{E}_{\mathrm{G}}$ such that*

$$M^{\mathrm{eas}} = s(\mathbf{trop}(M)) \quad \text{for all } M \subset \mathbb{T}.$$

Proof. Theorem 7.3.3 states that

$$I(M^{\mathrm{eas}}, P^{\mathrm{eas}}) = \mathscr{Y}\bigl(\mathbf{trop}(M), \mathbf{trop}(P)\bigr),$$
$$I(N^{\mathrm{eas}}, P^{\mathrm{eas}}) = \mathscr{Y}\bigl(\mathbf{trop}(N), \mathbf{trop}(P)\bigr).$$

Hence, if $\mathbf{trop}(M) = \mathbf{trop}(N)$, then the EA-sets M^{eas} and N^{eas} are numerically equivalent. Thus the set-theoretic map $s\colon \operatorname{Trop}_{\mathbb{T}} \to \mathscr{E}_{\mathrm{G}}$ exists. The fact that s is a ring homomorphism follows from (7.1). □

Corollary 7.3.10. *The ring of conditions $\mathscr{E}_G$ is generated by the images of exponential hypersurfaces.*

Proof. The ring of conditions of a torus is known to be generated by the numerical equivalence classes of algebraic hypersurfaces. It follows from Assertion 7.3.6 and Theorem 7.3.9 that $\mathscr{E}_G$ is a quotient ring of the ring of conditions $\mathscr{R}_{\mathbb{T}}$. If a graded ring is generated by the elements of degree 1, then its quotient rings inherit this property. □

Throughout the remainder of this subsection we rely on results from § 6. The formulations of the statements below are nonetheless fully self-contained.

We extend $\mathscr{Y}$ to the space of all tropical fans by setting it equal to zero on each homogeneous fan of codimension distinct from n. (We say that a k-dimensional fan of cones is homogeneous if each of its cones is a face of a k-dimensional cone.) On the space of tropical fans we consider the symmetric bilinear form

$$B_{\mathscr{Y}}(\mathscr{K}, \mathscr{L}) = \mathscr{Y}(\mathscr{K} \cdot \mathscr{L}).$$

The kernel $J_{\mathscr{Y}}$ of $B_{\mathscr{Y}}$ is an ideal of the ring $\operatorname{Trop}_{\mathbb{T}}$.

Corollary 7.3.11. *The ring of conditions $\mathscr{E}_G$ is isomorphic to the quotient ring $\operatorname{Trop}_{\mathbb{T}} / J_{\mathscr{Y}}$. In other words, the homomorphism s in Theorem 7.3.9 can be described as the factorization $\operatorname{Trop}_{\mathbb{T}} \to \operatorname{Trop}_{\mathbb{T}} / J_{\mathscr{Y}}$.*

Corollary 7.3.12. *The product operation in $\mathscr{E}_G$ defines a non-degenerate pairing $\mathscr{E}_{G,p} \times \mathscr{E}_{G,n-p} \to \mathbb{R}$.*

In conclusion, we present a geometric description of $\mathscr{E}_G$. Consider the Newton polytopes of exponential sums in E_G, that is, convex polytopes in $\operatorname{Re}(\mathbb{C}^n)^*$ with vertices at points in G. We denote by $\mathscr{H}$ the vector space of virtual convex polytopes generated by them. Let $S(\mathscr{H}) = \sum_{m\geqslant 0} S_m$ be the symmetric algebra of $\mathscr{H}$ and let I denote the linear functional on S_n whose value at the product of polytopes $\Lambda_1\cdots\Lambda_n$ is the mixed volume of the Λ_i. With I we associate the homogeneous ideal $J\subset S(\mathscr{H})$ generated by the following sets of generators: 1) $\ker I$, 2) $\sum_{m>n} S_m$, and 3) $\{s\in S_k \mid s\cdot S_{n-k}\subset \ker I,\ k=1,\dots,n-1\}$.

Theorem 7.3.13. *For an exponential sum $f\in E_G$ let Δ_f denote its Newton polytope and X_f the exponential hypersurface $f=0$. Then the correspondence $X_f\to\Delta_f$ extends to a ring isomorphism $\mathscr{E}_G\otimes_{\mathbb{Z}}\mathbb{R}\to S(\mathscr{H})/J$.*

7.4. The exponential BKK theorem. Theorem 7.4.1 below can be regarded as a tropical version of the BKK theorem (see Theorem 3.1.3), which arises in the context of quasi-algebraic exponential sums. It is used to derive the exponential BKK formula (Theorem 7.2.17). The proof of Theorem 7.4.1 reduces to using the tropical BKK theorem and Assertion 2.1.7. We skip the details of the proofs.

Consider the space H^* dual to H and endowed with the dual Euclidean metric. We identify it with $(\operatorname{Re}\mathfrak{T})^*/H^\perp$, where $(\operatorname{Re}\mathfrak{T})^*$ is the space of linear functionals on $\operatorname{Re}\mathfrak{T}$ and $H^\perp$ is the orthogonal complement to H. For a subset A of $(\operatorname{Re}\mathfrak{T})^*$ let A^H denote its projection onto the quotient $(\operatorname{Re}\mathfrak{T})^*/H^\perp$.

Theorem 7.4.1. *Let $\Delta_1,\dots,\Delta_n$ be convex polytopes in $(\operatorname{Re}\mathfrak{T})^*$ with vertices in the integer lattice and let $\mathscr{K}_1,\dots,\mathscr{K}_n\subset\operatorname{Re}\mathfrak{T}$ be the dual tropical fans of the Δ_i (see Definition 3.3.2). Then*

$$\mathscr{Y}(\mathscr{K}_1\cdots\mathscr{K}_n)=c(H)\operatorname{MV}(\Delta_1^H,\dots,\Delta_n^H),$$

where MV *is the mixed volume in H^* and $c(H)$ is a constant depending on the Euclidean space H.*

Theorem 7.4.1 yields the Kushnirenko–Bernstein formula for quasi-algebraic exponential sums (Theorem 7.2.17). Indeed, let $X_1,\dots,X_n$ be quasi-algebraic exponential hypersurfaces with equations $f_i=0$, and let M_i be models of the EA-sets X_i. These models are algebraic hypersurfaces. Let $\mathscr{K}_i$ be their tropicalizations. It follows from the above that

$$I(X_1,\dots,X_n)=\mathscr{Y}(\mathscr{K}_1\cdots\mathscr{K}_n).$$

Using Theorem 7.4.1, we obtain

$$I(X_1,\dots,X_n)=c(n)\operatorname{MV}(\Delta_1,\dots,\Delta_n),$$

where the Δ_i are the Newton polytopes of the exponential sums f_i. The value $n!/(2\pi)^n$ of the coefficient $c(n)$ is easy to find by looking at the simplest case where $f_i=\mathrm{e}^{z_i}-1$.

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